Smilansky-Solomyak model with a $\delta'$-interaction

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the usual way of constructing time-irreversible system is to couple the Hamiltonian with the bath of infinite degrees of freedom

but infinitely many degrees of freedom are not necessary

Uzy Smilansky proposed the model consisting of a quantum graph coupled with harmonic oscillators (a harmonic oscillator) and showed that if coupling is large enough, this system exhibits irreversible behaviour

simplest model: Schrödinger operator on a line coupled with a $\delta$ condition with a harmonic oscillator – largely studied

our model: Schrödinger operator on a line coupled with a $\delta'$ condition with a harmonic oscillator
Original Smilansky model

- the Hamiltonian formally written as

\[ H_\alpha = -\frac{\partial^2}{\partial x^2} + \frac{1}{2} \left( -\frac{\partial^2}{\partial y^2} + y^2 \right) + \alpha y \delta(x), \]

- precisely defined as a differential operator in \( L^2(\mathbb{R}^2) \)

\[ H_\alpha \psi = -\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} \left( -\frac{\partial^2 \psi}{\partial y^2} + y^2 \psi \right), \]

with the domain consisting of functions satisfying

\[ \frac{\partial \psi}{\partial x}(0^+, y) - \frac{\partial \psi}{\partial x}(0^-, y) = \alpha y \psi(0, y) \quad \text{for} \quad y \in \mathbb{R}. \]

- swap \( \alpha \to -\alpha \) is equivalent to the change \( y \to -y \) and hence it does not influence the spectrum, we can assume only \( \alpha > 0 \)
the continuous spectrum covers the interval \((1/2, \infty)\) for \(\alpha < \sqrt{2}\), covers the interval \((0, \infty)\) if \(\alpha = \sqrt{2}\) and the whole real axis if \(\alpha > \sqrt{2}\).

for \(\alpha \in (0, \sqrt{2})\) the discrete spectrum is nonempty, simple and is contained in \((0, 1/2)\); for \(\alpha > \sqrt{2}\) the point spectrum is empty.

the number of eigenvalues increases as \(\alpha \to \sqrt{2}\):

\[
N\left(\frac{1}{2}, H_\alpha\right) \sim \frac{1}{4} \sqrt{\frac{1}{\sqrt{2}(\sqrt{2} - \alpha)}}
\]

for \(\alpha\) large enough there is only one eigenvalue which behaves as

\[
\varepsilon_1(\alpha) = \frac{1}{2} - \frac{\alpha^4}{64} + \mathcal{O}(\alpha^5)
\]
Smilansky model with $\delta'$-interaction

- the Hamiltonian formally written as

$$H_\beta = -\frac{\partial^2}{\partial x^2} + \frac{1}{2} \left( -\frac{\partial^2}{\partial y^2} + y^2 \right) + \frac{\beta}{y} \delta'(x),$$

- precisely defined as a differential operator in $L^2(\mathbb{R}^2)$

$$H_\beta \psi(x, y) = -\frac{\partial^2 \psi}{\partial x^2}(x, y) + \frac{1}{2} \left( -\frac{\partial^2 \psi}{\partial y^2}(x, y) + y^2 \psi(x, y) \right)$$

with the domain consisting of functions in

$$\psi \in H^2((0, \infty) \times \mathbb{R}) \oplus H^2((\infty, 0) \times \mathbb{R})$$
satisfying

$$\psi(0+, y) - \psi(0-, y) = \frac{\beta}{y} \frac{\partial \psi}{\partial x}(0+, y), \quad (1)$$

$$\frac{\partial \psi}{\partial x}(0+, y) = \frac{\partial \psi}{\partial x}(0-, y). \quad (2)$$

- again, swap $\beta \rightarrow -\beta$ is equivalent to the change $y \rightarrow -y$ and hence it does not influence the spectrum, we can assume only $\beta > 0$
Spectral properties of the Smilansky model with $\delta'$

**Theorem 1 (absolutely continuous spectrum of the operators $H_0$ and $H_\beta$)**

The spectrum of operator $H_0$ is purely absolutely continuous, $\sigma(H_0) = [\frac{1}{2}, \infty)$ with $m_{ac}(E, H_0) = 2n$ for $E \in (n - \frac{1}{2}, n + \frac{1}{2})$, $n \in \mathbb{N}$.

For $\beta > 2\sqrt{2}$ the absolutely continuous spectrum of $H_\beta$ coincides with the spectrum of $H_0$. For $\beta \leq 2\sqrt{2}$ there is a new branch of continuous spectrum added to the spectrum of $H_0$. For $\beta = 2\sqrt{2}$ we have $\sigma(H_\beta) = [0, \infty)$ and for $\beta < 2\sqrt{2}$ the spectrum covers the whole real line.

- $m_{ac}$ denote the multiplicity function of the absolutely continuous spectra
Theorem 2 (discrete spectrum of the operator $H_\beta$ for $\beta \in (2\sqrt{2}, \infty)$)

Assume $\beta \in (2\sqrt{2}, \infty)$, then the discrete spectrum of $H_\beta$ is nonempty and lies in the interval $(0, \frac{1}{2})$. The number of eigenvalues is approximately given by

$$\frac{1}{4\sqrt{2} \left( \frac{\beta}{2\sqrt{2}} - 1 \right)}$$

as $\beta \to 2\sqrt{2} +$.

Theorem 3 (discrete spectrum of the operator $H_\beta$ for large $\beta$)

For large enough $\beta$ there is a single eigenvalue which asymptotically behaves as

$$\Lambda_1 = \frac{1}{2} - \frac{4}{\beta^4} + O(\beta^{-5})$$
The quadratic form

- the quadratic form $a_\beta[\psi] = a_0[\psi] + \frac{1}{\beta} b[\psi]$

$$a_0[\psi] = \int_{\mathbb{R}^2} \left( \left| \frac{\partial \psi}{\partial x} \right|^2 + \frac{1}{2} \left| \frac{\partial \psi}{\partial y} \right|^2 + \frac{1}{2} y^2 |\psi|^2 \right) \, dx \, dy,$$

$$b[\psi] = \int_{\mathbb{R}} y \left| \psi(0+, y) - \psi(0-, y) \right|^2 \, dy$$

is associated with the operator $H_\beta$. The domain $D = \text{dom } a_0$ of the form $a_0$ is

$$D = \{ \psi \in H^1((0, \infty) \times \mathbb{R}) \oplus H^1((-\infty, 0) \times \mathbb{R}) ; a_0[\psi] < \infty \}$$
Bound on the quadratic form

Theorem 4

If $\beta \geq 2\sqrt{2}$ it holds

$$a_\beta[\Psi] \geq \frac{1}{2} \left( 1 - \frac{2\sqrt{2}}{\beta} \right) \|\Psi\|^2.$$

Lemma 5

For complex numbers $c, d$ it holds

$$2|\text{Re}(\overline{c}d)| \leq |c|^2 + |d|^2.$$
Lemma 6

It holds

\[ \gamma (|\psi(0+)|^2 + |\psi(0-)|^2) \leq \int_{\mathbb{R}} (|\psi'(x)|^2 + \gamma^2 |\psi(x)|^2) \, dx \]

\[ \forall \psi \in H^1((0, \infty)) \oplus H^1((\infty, 0)), \quad \gamma > 0, \]

with the equality attained on the subspace generated by

\[ \tilde{\psi}_\gamma(x) = \frac{\text{sgn} \, x}{\sqrt{2\gamma}} \, e^{-\gamma|x|}, \quad \gamma > 0. \]
Proof of Lemma 6.

we have

$$0 \leq \int_{-\infty}^{0} |\psi'(x) - \gamma \psi(x)|^2 \, dx = \int_{-\infty}^{0} (|\psi'(x)|^2 + \gamma^2 |\psi(x)|^2) \, dx$$

$$- \gamma \int_{-\infty}^{0} (\bar{\psi}'(x)\psi(x) + \psi'(x)\bar{\psi}(x)) \, dx$$

$$= \int_{-\infty}^{0} (|\psi'(x)|^2 + \gamma^2 |\psi(x)|^2) \, dx - \gamma [|\psi(x)|^2]_{-\infty}^{0},$$

and therefore

$$\int_{-\infty}^{0} (|\psi'(x)|^2 + \gamma^2 |\psi(x)|^2) \, dx \geq \gamma |\psi(0-)|^2.$$
Proof of Theorem 4.

- we use separation of variables and the expansion of $\Psi$ in the harmonic oscillator basis, i.e. Hermite functions in the variable $y$ normalized in $L^2(\mathbb{R})$

$$\Psi(x, y) = \sum_{n \in \mathbb{N}_0} \psi_n(x) \chi_n(y),$$  \hspace{1cm} (4)

- we insert this into the form $a_0$

$$a_0[\Psi] = \sum_{n \in \mathbb{N}_0} \int_{\mathbb{R}} \left( |\psi'_n(x)|^2 + \left( n + \frac{1}{2} \right) |\psi_n(x)|^2 \right) \, dx \hspace{1cm} (5)$$

- we use twice expansion (4) and the relation

$$\sqrt{n+1} \chi_{n+1}(y) - \sqrt{2} y \chi_n(y) + \sqrt{n} \chi_{n-1}(y) = 0, \quad n \in \mathbb{N}_0,$$  \hspace{1cm} (6)
Proof of Theorem 4 (continued).

we obtain

\[ b[\Psi] = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \sum_{m \in \mathbb{N}_0} \sum_{n \in \mathbb{N}_0} (\tilde{\psi}_m(0+) - \tilde{\psi}_m(0-))\tilde{\chi}_m(y) \]

\[ (\psi_n(0+) - \psi_n(0-)) \left[ \sqrt{n+1}\chi_{n+1}(y) + \sqrt{n}\chi_{n-1}(y) \right] \, dy \]

\[ = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{N}_0} \left[ ((\tilde{\psi}_{n+1}(0+) - \tilde{\psi}_{n+1}(0-))\sqrt{n+1} \right. \]

\[ + (\tilde{\psi}_{n-1}(0+) - \tilde{\psi}_{n-1}(0-))\sqrt{n}] (\psi_n(0+) - \psi_n(0-)) = \]

\[ = \frac{2}{\sqrt{2}} \sum_{n \in \mathbb{N}} \sqrt{n} \text{Re} [((\tilde{\psi}_n(0+) - \tilde{\psi}_n(0-))(\psi_{n-1}(0+) - \psi_{n-1}(0-))] . \]

(7)

we employed the Hermite functions orthonormality here and in the last line we have changed the summation index, \( n + 1 \rightarrow n \), in the first part of the sum.
Proof of Theorem 4 (continued).

- it follows from Lemma 5 that

$$|b[\Psi]| \leq \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{N}} \sqrt{n}|\psi_n(0^+) - \psi_n(0^-)|^2 +$$

$$+ |\psi_{n-1}(0^+) - \psi_{n-1}(0^-)|^2).$$

- changing the summation index in the second part of the sum we get

$$|b[\Psi]| \leq \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{N}_0} \sqrt{n + 1}|\psi_n(0^+) - \psi_n(0^-)|^2$$

$$\leq \sum_{n \in \mathbb{N}_0} \sqrt{2n + 1}|\psi_n(0^+) - \psi_n(0^-)|^2,$$

where we have used the inequality

$$\sqrt{n + \sqrt{n + 1}} < \sqrt{2(2n + 1)}$$
using subsequently Lemmata 5 and 6 we obtain

\[ |b[\Psi]| \leq 2\sqrt{2} \sum_{n \in \mathbb{N}_0} \sqrt{n + \frac{1}{2}} \left( |\psi_n(0+)|^2 + |\psi_n(0-)|^2 \right) \]

\[ \leq 2\sqrt{2} \sum_{n \in \mathbb{N}_0} \int_{\mathbb{R}} \left( |\psi'_n(x)|^2 + \left( n + \frac{1}{2} \right) |\psi_n(x)|^2 \right) dx \]

\[ = 2\sqrt{2} a_0[\Psi]. \]

we use \( a_0[\Psi] \geq \frac{1}{2} \|\Psi\|^2 \), which follows from (5)

we obtain

\[ a_\beta[\Psi] = a_0[\Psi] + \frac{1}{\beta} b[\Psi] \geq \left( 1 - \frac{2\sqrt{2}}{\beta} \right) a_0[\Psi] \]

\[ \geq \frac{1}{2} \left( 1 - \frac{2\sqrt{2}}{\beta} \right) \|\Psi\|^2, \]
Construction of the Jacobi operator

- we will rephrase the problem with using certain Jacobi operator
- we substitute to eq. (1) the Ansatz (4) for $\Psi$, multiply the equation by $\bar{\chi}_m(y)$, integrate with respect to $y$ over $\mathbb{R}$, and use the orthonormality

$$
\sum_{n \in \mathbb{N}_0} \int_{\mathbb{R}} \bar{\chi}_m(y) y(\psi_n(0+) - \psi_n(0-))\chi_n(y) \, dy = \\
= \beta \sum_{n \in \mathbb{N}_0} \int_{\mathbb{R}} \frac{\partial \psi_n}{\partial x}(0+) \bar{\chi}_m(y)\chi_n(y) \, dy
$$
relation (6) then yields the condition

$$\beta \frac{\partial \psi_m}{\partial x}(0+) = \sum_{n \in \mathbb{N}_0} \frac{1}{\sqrt{2}} \int_{\mathbb{R}} (\psi_n(0+) - \psi_n(0-)) \bar{\chi}_m(y) \left(\sqrt{n+1} \chi_{n+1}(y) + \sqrt{n} \chi_{n-1}(y)\right) \, dy$$

$$= \frac{\sqrt{m}}{\sqrt{2}} (\psi_{m-1}(0+) - \psi_{m-1}(0-))$$

$$+ \frac{\sqrt{m+1}}{\sqrt{2}} (\psi_{m+1}(0+) - \psi_{m+1}(0-)), \quad (8)$$

on the other hand, the condition (2) implies

$$\frac{\partial \psi_n}{\partial x}(0+) = \frac{\partial \psi_n}{\partial x}(0-) \quad (9)$$
consider now the eigenvalue problem for the operator $H_\beta$, which is equivalent to the set of equations

$$-\phi''_n(x) + \left(n + \frac{1}{2} - \Lambda\right)\phi_n(x) = 0, \quad x = 0, \quad n \in \mathbb{N}_0 \quad (10)$$

under the matching conditions (8) and (9), for $\phi_n \upharpoonright \mathbb{R}_\pm \in H^2(\mathbb{R}_\pm)$ where $\Lambda$ is the sought eigenvalue.

we define $\zeta_n(\Lambda) = \sqrt{n + \frac{1}{2} - \Lambda}$ taking the branch of the square root which is analytic in $\mathbb{C}\setminus[n + \frac{1}{2}, \infty)$ and for number $\Lambda$ from this set it holds

$$\Re \zeta_n(\Lambda) > 0, \quad \Im \zeta_n(\Lambda) \cdot \Im \Lambda < 0.$$

solutions to the equation (10) in $L^2(\mathbb{R}_\pm)$ are

$$\phi_n(x, \Lambda) = k_1(\Lambda) e^{-\zeta_n(\Lambda)x}, \quad x > 0,$$

$$\phi_n(x, \Lambda) = k_2(\Lambda) e^{\zeta_n(\Lambda)x}, \quad x < 0,$$

where from (9) we have $k_1(\Lambda) = -k_2(\Lambda)$.
we use the normalization \( \phi_n(x, \Lambda) = C_n \eta_n(x, \Lambda) \) with
\[
\eta_n(x, \Lambda) := \pm \left( n + \frac{1}{2} \right)^{1/4} e^{\mp \zeta_n(\Lambda) x} \cdot \ x \in \mathbb{R}_\pm.
\]

hence
\[
\phi_n(0^+, \Lambda) - \phi_n(0^-, \Lambda) = 2 C_n \left( n + \frac{1}{2} \right)^{1/4} ,
\]
\[
\frac{\partial \phi_n}{\partial x}(0^+, \Lambda) = - C_n \left( n + \frac{1}{2} \right)^{1/4} \zeta_n(\Lambda) \cdot \quad (11)
\]

substituting from here to eq. (8) we obtain the relation
\[
(n + 1)^{1/2} \left( n + \frac{3}{2} \right)^{1/4} C_{n+1} + 2 \mu \left( n + \frac{1}{2} \right)^{1/4} \zeta_n(\Lambda) C_n + \\
n^{1/2} \left( n - \frac{1}{2} \right)^{1/4} C_{n-1} = 0 , \quad n \in \mathbb{N}_0 \quad (12)
\]

with \( \mu := \frac{\beta}{2\sqrt{2}} \)

this equation defines the same Jacobi operator \( \mathcal{J}(\Lambda, \mu) \) as for \( \delta \)-condition, only our parameter \( \mu \) differs
Absolutely continuous spectrum of $H_\beta$

- one can represent the resolvent using the Krein formula with the obtained Jacobi operator
- one can proceed similarly to the known case of $\delta$-condition and prove the following theorems
Theorem 7 (absolutely continuous spectrum of $H_\beta$)

\[
\sigma_{ac}(H_\beta) = \sigma_{ac}(H_0) \cup \sigma_{ac}(J_0(\beta/(2\sqrt{2}))),
\]
\[
m_{ac}(E, H_\beta) = m_{ac}(E, H_0) + m_{ac}(E, J_0(\beta/(2\sqrt{2}))).
\]

where

\[
J_0(\mu) := DS + S^*D + 2\mu Y_0
\]

with

\[
D, S : \ell^2(\mathbb{N}_0) \mapsto \ell^2(\mathbb{N}_0),
\]
\[
D\{\omega_n\} : \{r_0, r_1, \ldots\} \mapsto \{\omega r_0, \omega_1 r_1, \ldots\},
\]
\[
D := D(d_n), \quad Y_0 := D\{n + 1/2\},
\]
\[
S : \{r_0, r_1, \ldots\} \mapsto \{0, r_0, r_1, \ldots\},
\]
\[
d_n := n^{1/2}(n + \frac{1}{2})^{1/4}(n - \frac{1}{2})^{1/4}.
\]
Theorem 8 (spectrum of $\mathcal{J}_0$)

$$\sigma(\mathcal{J}_0(\mu)) = (-\infty, \infty) \quad \text{for} \quad \mu < 1,$$

$$\sigma(\mathcal{J}_0(1)) = [0, \infty),$$

$$\sigma_{ac}(\mathcal{J}_0(\mu)) = \emptyset \quad \text{for} \quad \mu > 1,$$

$$m_{ac}(E, \mathcal{J}_0(\mu)) = 1 \quad \text{a.e. on} \quad \sigma(\mathcal{J}_0(\mu)).$$

Since we have $\mu = \frac{\beta}{2\sqrt{2}}$, these two theorem in combination with the well-known spectrum of $\mathbf{H}_0$ prove the claim of Theorem 1.
Discrete spectrum of $H_\beta$

**Proof of Theorem 3.**

- first we check that the spectrum on $(-\infty, \frac{1}{2})$ is non-empty using a variational argument.
- the idea is to construct an element $\Psi^\varepsilon \in D$ such that $a_\beta[\Psi^\varepsilon] < \frac{1}{2} \|\Psi^\varepsilon\|^2$
- consider functions $\psi_0, \psi_1$ satisfying the conditions

$$\psi_0(0+) - \psi_0(0-) = -C < 0, \quad \psi_1(0+) - \psi_1(0-) = 1,$$

and such that $\Psi = \{\psi_0, \psi_1, 0, 0, \ldots\} \in D$
- we scale the first one, $\psi_0^\varepsilon(x) := \psi_0(\varepsilon x)$, and put $\Psi^\varepsilon := \{\psi_0^\varepsilon, \psi_1, 0, 0, \ldots\}$ which belongs again to $D$
Proof of Theorem 3 (continued).

- from (5) and (7) we have

\[ a_\beta[\Psi^\varepsilon] - \frac{1}{2}\|\Psi^\varepsilon\|^2 = \int_{\mathbb{R}} (|\psi_0'(x)|^2 + |\psi_1(x)|^2 + |\psi_1'(x)|^2) \, dx - \frac{\sqrt{2}}{\beta} C = \]

\[ = \int_{\mathbb{R}} (\varepsilon|\psi_0'(x)|^2 + |\psi_1(x)|^2 + |\psi_1'(x)|^2) \, dx - \frac{\sqrt{2}}{\beta} C \]

- choosing \( \varepsilon \) small enough and \( C \) large enough one can achieve that the right-hand side of the last equation is negative, which means that the spectrum below \( \frac{1}{2} \) is nonempty for any \( \beta > 0 \)
Proof of Theorem 3 (continued).

- one can proof that $N_-(\frac{1}{2}, H_\beta) = N_+(\mu, J_0)$ or $N_-(\frac{1}{2}, H_\beta) = N_+(\mu, J_0) + 1$, where $N_-$ ($N_+$ is the number of eigenvalues below (above) certain value)

- using the fact on the previous slide, and the fact that the eigenvalues of $J_0$ have a single accumulation point at 1 (and consequently, there is a $\mu$ such that there are no eigenvalues of $J_0$ larger than $\mu$) we find that for $\beta$ large enough the operator $H_\beta$ has exactly one simple eigenvalue.

- the asymptotic expansion of this eigenvalue $\Lambda_1$ can be found by an argument similar to the original Smilansky model
Proof of Theorem 3 (continued).

- the system of equations (12) can be after substitution $Q_n = (n + \frac{1}{2})^{1/4} C_n$ rewritten as

$$Q_1 + 2\mu \sqrt{\frac{1}{2} - \Lambda_1} Q_0 = 0, \quad (13)$$

$$(n + 1)^{1/2} Q_{n+1} + 2\mu \zeta_n(\Lambda_1) Q_n + n^{1/2} Q_{n-1} = 0,$$

$$n \in \mathbb{N}. \quad (14)$$

- we normalize $\|Q\| := \sum_{n=0}^{\infty} |Q_n|^2 = 1$, using then

$$\sqrt{n} \leq \sqrt{n + \frac{1}{2} - \Lambda_1} = \zeta_n(\Lambda_1) \text{ and } \sqrt{n + 1} \leq \sqrt{2(n + \frac{1}{2} - \Lambda_1)}$$

we obtain from (14) the estimate

$$|Q_n| \leq \frac{1}{2\mu} |Q_{n-1}| + \frac{1}{\sqrt{2\mu}} |Q_{n+1}|. \quad (15)$$
Proof of Theorem 3 (continued).

- in the analogy with Lemma 5 we have
  \[ |Q_n|^2 \leq \frac{1}{2\mu^2} |Q_{n-1}|^2 + \frac{1}{\mu^2} |Q_{n+1}|^2, \]
  hence
  \[ \sum_{n=1}^{\infty} |Q_n|^2 \leq \frac{1}{2\mu^2} \sum_{n=0}^{\infty} |Q_n|^2 + \frac{1}{\mu^2} \sum_{n=2}^{\infty} |Q_n|^2 \leq \frac{3}{2\mu^2}, \]
  where we have used the mentioned normalization

- from here it follows that
  \[ |Q_0| = \left( \sum_{n=0}^{\infty} |Q_n|^2 - \sum_{n=1}^{\infty} |Q_n|^2 \right)^{1/2} = 1 + O\left(\mu^{-2}\right). \quad (16) \]

- without loss of generality we may suppose that \( Q_0 \) is positive
Proof of Theorem 3 (continued).

- from (15) with $n = 2$ with the use of the normalization we obtain

$$|Q_2| \leq \frac{1}{2\mu} + \frac{1}{\sqrt{2}\mu} \quad \Rightarrow \quad Q_2 = \mathcal{O} \left( \mu^{-1} \right).$$

- furthermore, from (14) and (16) we get

$$Q_1 = \frac{1}{2\mu} + \mathcal{O} \left( \mu^{-2} \right).$$

- from (13) we obtain

$$(\frac{1}{2} - \Lambda_1)^{1/2} = - \frac{Q_1}{2\mu Q_0} = - \frac{1}{4\mu^2} + \mathcal{O} \left( \mu^{-3} \right),$$

or equivalently

$$\frac{1}{2} - \Lambda_1 = \frac{1}{16\mu^4} + \mathcal{O} \left( \mu^{-5} \right) = \frac{4}{\beta^4} + \mathcal{O} \left( \beta^{-5} \right),$$

which concludes the proof.

Thank you for your attention!

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