Eigenvalue asymptotics for the damped wave equation on metric graphs

Jiří Lipovský

University of Hradec Králové, Faculty of Science
jiri.lipovsky@uhk.cz

joint work with P. Freitas

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Outline

1. Description of the model
2. Location of the eigenvalues and high frequency absissas
3. Pseudoorbit expansion
4. Number of high frequency abscissas
5. Examples
6. Conclusion
Description of the model

- metric graph with $N$ finite edges
- on each edge damped wave equation

$$\partial_{tt} w_j(t, x) + 2a_j(x) \partial_t w_j(t, x) = \partial_{xx} w_j(t, x) + b_j(x) w_j(t, x)$$

- for constant $a_j$, $b_j$ the ansatz $w_j(t, x) = e^{\lambda t} u_j(x)$ allows us to rewrite equation it as

$$\partial_{xx} u_j(x) - (\lambda^2 + 2\lambda a_j - b_j) u_j(x) = 0$$

- we define $\tilde{\lambda}_j(\lambda) := \sqrt{\lambda^2 + 2\lambda a_j - b_j}$ and obtain solutions $e^{\pm \tilde{\lambda}_j x}$

- coupling conditions at the vertices

$$(U - I) \psi + i(U + I) \psi' = 0$$

- substituting for $\psi$ and $\psi'$ one obtains the secular equation
alternatively, eigenvalues of the operator

\[ H = \begin{pmatrix}
0 & I \\
I \frac{d^2}{dx^2} + B & -2A
\end{pmatrix}, \]

with diagonal \( N \times N \) matrices \( A \) and \( B \)

the domain of the operator consists of functions \((\psi_1(x), \psi_2(x))^T\) with components of both \( \psi_1 \) and \( \psi_2 \) in \( W^{2,2}(e_j) \) and satisfying the coupling conditions

\[(U - I) \psi + i(U + I) \psi' = 0,\]

at the vertices
Definition of high frequency abscissa

we say that $c_0$ is a high frequency abscissa of the operator $H$ if there exists a sequence of eigenvalues of $H$, say $\{\lambda_n\}_{n=1}^{\infty}$, such that

$$\lim_{n \to \infty} \text{Im} \lambda_n = \pm \infty \text{ and } \lim_{n \to \infty} \text{Re} \lambda_n = c_0.$$
Location of eigenvalues and high frequency abscissas

**Theorem**

If $\lambda$ is an eigenvalue of $H$ with nontrivial imaginary part $\Im(\lambda) \neq 0$, then its real part satisfies

$$
\Re(\lambda) = -\frac{\sum_{j=1}^{N} \int_{0}^{l_j} a_j(x)|u_j(x)|^2 \, dx}{\sum_{j=1}^{N} \|u_j(x)\|_2^2},
$$

where $u_j(x)$ denotes the corresponding wavefunction components.

**Corollary**

Let us consider a damped wave equation on the graph $\Gamma$ with damping functions on the edges $a_j(x)$ and potentials $b_j(x)$. Denote the average of the damping function on each edge by $\bar{a}_j$. Then all high frequency abscissas lie in the interval $[-\max_j \bar{a}_j, -\min_j \bar{a}_j]$. 
Idea of the proof

- we multiply $\partial_{xx} u_j(x) - (\lambda^2 + 2\lambda a_j(x) - b_j(x)) u_j(x) = 0$ by $\bar{u}_j(x)$ and integrate over the $j$-th edge

$$0 = \int_0^{l_j} \bar{u}_j(x) u''_j(x) \, dx - \int_0^{l_j} [\lambda^2 + 2a_j(x)\lambda - b_j(x)] |u_j(x)|^2 \, dx =$$

$$= - \int_0^{l_j} |u'_j(x)|^2 \, dx + [\bar{u}_j(x) u'_j(x)]_0^{l_j} -$$

$$- \int_0^{l_j} [\lambda^2 + 2a_j(x)\lambda - b_j(x)] |u_j(x)|^2 \, dx .$$

- we sum over all the edges
- we prove that the term coming from the coupling conditions is real
- for the imaginary part of previous equation we have

$$0 = 2i \Im(\lambda) \sum_{j=1}^{N} \int_0^{l_j} (a_j(x) + \Re(\lambda)) |u_j(x)|^2 \, dx$$
Location of high frequency abscissas

**Theorem**

Let $\Gamma$ be a graph with $N$ edges of commensurate lengths $l_j = m_j l_0$, $m_j \in \mathbb{N}$, $j = 1, \ldots, N$, with the coupling conditions given by the coupling matrix $U$. Let the damping functions $a_j(x)$ and $b_j(x)$ be bounded and continuous on each edge. Let $\lambda_n$ be eigenvalues of $H$ and $\mu_n$ eigenvalues for $a_j$ and $b_j$ replaced by their averages. Then the constant terms $c_0$ in the asymptotic expansion of $\lambda_n$ coincide with the corresponding constant terms in the asymptotic expansion of $\mu_n$.

**Theorem**

Let $\Gamma$ be a graph with standard coupling which contains a loop of $N$ edges of length one with averages of damping coefficients on all of the edges of the loop equal to $a$. Then there is a high frequency abscissa at $-a$. 
Idea of the proof

- due to the previous theorem one can assume constant damping equal to $a$ at the loop
- we construct an eigenfunction of $H$ supported only on the loop with zeros at the vertices
- using the ansatz $f_j(x) = \alpha_j(e^{\tilde{\lambda}_j(\lambda)x} - e^{-\tilde{\lambda}_j(\lambda)x})$ with $\tilde{\lambda}_j(\lambda) = \sqrt{\lambda^2 + 2a_j\lambda - b_j}$ on the $j$-th edge we have

$$f_j(0) = 0, \quad f_j(1) = \alpha_j(e^{\tilde{\lambda}_j(\lambda)} - e^{-\tilde{\lambda}_j(\lambda)}),$$
$$f_j'(0) = 2\alpha_j\tilde{\lambda}_j(\lambda), \quad f_j'(1) = \alpha_j\tilde{\lambda}_j(\lambda)(e^{\tilde{\lambda}_j(\lambda)} + e^{-\tilde{\lambda}_j(\lambda)}).$$

- the continuity of the derivatives leads to $\alpha_{j+1} - \alpha_j(-1)^n = 0$ and thus $(-1)^{nN} = 1$
- for $N$ even one has two sequences with $c_0^{(1)} = -a$ and $c_0^{(2)} = -a + \pi i$, while for $N$ odd there is a sequence with $c_0 = -a$. 
Pseudoorbit expansion


- graph $\Gamma$ replaced by a directed graph $\Gamma_2$

- ansatz

$$f_{e_j}(x) = \alpha_{e_j}^{\text{in}} e^{\tilde{\lambda}_j x} + \alpha_{e_j}^{\text{out}} e^{-\tilde{\lambda}_j x},$$

$$f_{\hat{e}_j}(x) = \alpha_{\hat{e}_j}^{\text{in}} e^{\tilde{\lambda}_j x} + \alpha_{\hat{e}_j}^{\text{out}} e^{-\tilde{\lambda}_j x}$$

- due to the relation $f_{e_j}(x) = f_{\hat{e}_j}(l_j - x)$ we have

$$\alpha_{\hat{e}_j}^{\text{out}} = e^{\tilde{\lambda}_j l_j} \alpha_{e_j}^{\text{in}}, \quad \alpha_{e_j}^{\text{out}} = e^{\tilde{\lambda}_j l_j} \alpha_{\hat{e}_j}^{\text{in}}.$$
we define the vertex scattering matrix $\sigma_V(\lambda)$ by
\[ \vec{\alpha}_{\text{out}}^v = \sigma_V(\lambda) \vec{\alpha}_{\text{in}}^v \quad \text{with} \quad \vec{\alpha}^\text{in, out}^v = (\alpha_{e_{v1}}^\text{in, out}, \ldots, \alpha_{e_{vd}}^\text{in, out})^T. \]

we define $\Sigma(\lambda)$ as a block-diagonalizable matrix written in the basis corresponding to
\[ \vec{\alpha} = (\alpha_{e_1}, \ldots, \alpha_{e_N}, \alpha_{\hat{e}_1}, \ldots, \alpha_{\hat{e}_N})^T, \]
which is block diagonal with blocks $\sigma_V(\lambda)$ if transformed to the basis
\[ (\alpha_{e_{v11}}^\text{in}, \ldots, \alpha_{e_{vd1}}^\text{in}, \alpha_{e_{v21}}^\text{in}, \ldots, \alpha_{e_{vd2}}^\text{in}, \ldots, )^T. \]

we define
\[ J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \]
scattering matrix $S = J \Sigma$ and
\[ L = \exp \left( \text{diag} \left( -\tilde{\lambda}_1 l_1, \ldots, -\tilde{\lambda}_N l_N, -\tilde{\lambda}_1 l_1, \ldots, -\tilde{\lambda}_N l_N \right) \right). \]
we obtain
\[
\begin{pmatrix}
\vec{\alpha}_{in}^e \\
\vec{\alpha}_{in}^\hat{e}
\end{pmatrix} = L \begin{pmatrix}
\vec{\alpha}_{out}^e \\
\vec{\alpha}_{out}^\hat{e}
\end{pmatrix} = LJ \begin{pmatrix}
\vec{\alpha}_{out}^e \\
\vec{\alpha}_{out}^\hat{e}
\end{pmatrix} = LJ\Sigma(\lambda) \begin{pmatrix}
\vec{\alpha}_{in}^e \\
\vec{\alpha}_{in}^\hat{e}
\end{pmatrix}
\]

the secular equation becomes
\[
\det \left( LJ\Sigma(\lambda) - I_{2N \times 2N} \right) = 0.
\]

**Theorem**

*Let us assume a vertex coupling matrix*

\[
U = V^{-1} \begin{pmatrix}
-I_{n-} & 0 & 0 \\
0 & I_{n+} & 0 \\
0 & 0 & D
\end{pmatrix} V
\]

*with V unitary and D unitary and diagonal. Then*

\[
\sigma(\lambda) = V^{-1} \begin{pmatrix}
-I_{n-} & 0 & 0 \\
0 & I_{n+} & 0 \\
0 & 0 & I_{d-n- -n+}
\end{pmatrix} V + O(1/n).
\]
a periodic orbit is a closed trajectory on the graph $\Gamma_2$

an irreducible pseudo orbit $\tilde{\gamma}$ is a collection of periodic orbits where none of the directed bonds is contained more than once

let $m_{\tilde{\gamma}}$ denote the number of periodic orbits in $\tilde{\gamma}$,

$L_{\tilde{\gamma}} = \sum_{e \in \tilde{\gamma}} \tilde{\lambda}_e l_e$ where the sum is over all directed bonds in $\tilde{\gamma}$

the coefficients $A_{\tilde{\gamma}} = \prod_{\gamma_j \in \tilde{\gamma}} A_{\gamma_j}$ with $A_{\gamma_j}$ given as multiplication of entries of $S(\lambda)$ along the trajectory $\gamma_j$.

**Theorem**

The secular equation for the damped wave equation on a metric graph is given by

$$\sum_{\tilde{\gamma}} (-1)^{m_{\tilde{\gamma}}} A_{\tilde{\gamma}}(\lambda) \exp(-L_{\tilde{\gamma}}(\lambda)) = 0$$

with $L_{\tilde{\gamma}}$ being the sum of the lengths of all directed edges along a particular irreducible pseudo orbit $\tilde{\gamma}$. 
Theorem

Let $\Gamma$ be an equilateral graph with $N$ edges of the length 1. Let us assume a damped wave equation on $\Gamma$ with damping and potential functions constant on each edge $a_j(x) \equiv a_j$, $b_j(x) \equiv b_j$ and with general coupling for a given unitary matrix $U$. Then there exist numbers $n_0 \in \mathbb{N}$, $c_0^{(s)} \in \mathbb{C}$, $s = 1, \ldots, 2N$ and $c_1 \in \mathbb{R}$ such that for every $n \geq n_0$ all eigenvalues of $H$ are within the following set

$$\{ \lambda, |\text{Im} \lambda| \leq 2\pi n_0 \} \cup \bigcup_{s=1}^{2N} \bigcup_{n=n_0}^{\infty} B \left( 2\pi n i + c_0^{(s)}, \frac{c_1}{n} \right),$$

where $B(x_0, r)$ denotes the circle in the complex plane with center $x_0$ and radius $r$. 
Idea of the proof

- Eigenvalues are given by $\lambda_{ns} = 2\pi in + c_0^{(s)} + O(1/n)$.
- Since the first term of the $n$ asymptotics of the secular equation uniquely determines the coefficients $c_0$, one can use the pseudo orbit expansion with the first term of $n$ asymptotics of vertex scattering matrices.
- Since the directed graph corresponding to $\Gamma$ has $2N$ edges, we obtain a polynomial equation in $y = e^{c_0}$ of $2N$-th order.
- It has $2N$ roots uniquely determining the values of $c_0^{(s)}$. 
Number of distinct high frequency abscissas – better upper bound

Theorem

Let $\Gamma$ be a graph with $N$ edges all of which have lengths equal to 1, (general) Robin coupling at the boundary and standard coupling otherwise. Let us suppose that the graph is bipartite, i.e. it does not have any closed loop of odd length (there is no such a sequence of odd number of edges in which the $m$-th and $(m+1)$-th edges, including both the last and first one, have a joint vertex). Then for any damping functions bounded and $C^2$ at each edge there are at most $N$ high frequency abscissas.
Idea of the proof

- all closed orbits have an even number of edges
- in the first term of the $n$ asymptotics of the secular equation there are only terms with $e^{2c_0}$
- polynomial equation in $e^{2c_0}$ of $N$-th order – at most $N$ roots
Theorem

Let $\Gamma$ be a tree graph with $N$ edges all with unit length, Robin coupling at the boundary and standard coupling otherwise. Let us suppose that all vertices have odd degree. Then there always exists such a damping for which the number of high frequency abscissas is greater than or equal to $N$. 
Idea of the proof

**Lemma**

Let $\Gamma$ be a tree graph with standard coupling and all edges of length one, $\Gamma_2$ the corresponding oriented graph. Let $\{e_1, \ldots, e_{2N}\}$ be a set of edges on $\Gamma$, $\tilde{\gamma}$ a pseudo orbit on $\{e_1, \ldots, e_{2N}, \hat{e}_1, \ldots, \hat{e}_{2N}\} \subset \Gamma_2$ and $\mathcal{X}$ be a vertex of $\Gamma$ of degree $d$ and let there be $v$ edges emanating from $\mathcal{X}$ denoted by $\{e_1, \ldots, e_v\}$. Let $s_1 = \frac{2}{d} - 1$, $s_2 = \frac{2}{d}$ be on-diagonal and off-diagonal elements of the scattering matrix at $\mathcal{X}$, respectively. For a particular pseudo orbit $\tilde{\gamma}'$ let $\Gamma_3(\tilde{\gamma}')$ be a collection of all pseudo orbits which can be obtained from $\tilde{\gamma}'$ by all possible changes at $\mathcal{X}$. Then the coefficient in $\sum_{\tilde{\gamma} \in \Gamma_3(\tilde{\gamma}')} A_{\tilde{\gamma}}(\lambda)$ corresponding to the vertex $\mathcal{X}$ is $A_{\mathcal{X}} = s_1^v (s - 1)^{v-1}[(v - 1)s + 1]$ with $s = \frac{s_2}{s_1} = \frac{2}{2-d}$. 
let us assume a given set of the edges of the graph $\Gamma$ and all pseudo orbits which go through every edge of this set twice, then the contribution of all these pseudo orbits cancels if and only if there is a vertex of $\Gamma$ of degree $d = 2v$ and the above set of edges contains $v$ edges emanating from this vertex.

this follows from the previous lemma, since

$$d = 2 - \frac{2}{s} = 2 + 2(v - 1) = 2v$$

for $s = -\frac{1}{v-1}$

the first term of the $n$ expansion of the secular equation can be written as (for simplicity we omit $n$ to the corresponding power)

$$C_N e^{2a_1 + 2a_2 + \cdots + 2a_N} y^N + C_{N-1} e^{2a_1 + 2a_2 + \cdots + 2a_{N-1}} [1 +$$

$$+ O \left( e^{-2(a_{N-1} - a_N)} \right) ] y^{N-1} + \cdots +$$

$$+ C_2 e^{2a_1 + 2a_2} \left[ 1 + O \left( e^{-2(a_2 - a_3)} \right) \right] y^2 +$$

$$+ C_1 e^{2a_1} \left[ 1 + O \left( e^{-2(a_1 - a_2)} \right) \right] y + C_0 = 0,$$
we use $0 \ll a_N \ll a_{N-1} \ll \cdots \ll a_1$

for $y$ being close to $e^{-2a_1}$ the last two terms are dominant

hence for the roots of the previous polynomial equation of the $N$-th order we get

$$y_j = -\frac{C_{j-1}}{C_j} e^{-2a_j} \left[ 1 + O \left( e^{-2(a_j-a_{j+1})} \right) \right] ,$$
Example – two loops with different damping coefficients

\[ a_2 \quad a_1 \]
\[ a_2 \quad a_1 \]

- secular equation

\[
\sinh \frac{3}{2} \tilde{\lambda}_1 \sinh \frac{3}{2} \tilde{\lambda}_2 \\
\left[ (\tilde{\lambda}_1 + \tilde{\lambda}_2) \sinh \frac{3(\tilde{\lambda}_1 + \tilde{\lambda}_2)}{2} + (\tilde{\lambda}_1 - \tilde{\lambda}_2) \sinh \frac{3(\tilde{\lambda}_1 - \tilde{\lambda}_2)}{2} \right] = 0.
\]

- using the asymptotic expansion \( \lambda_n = 2\pi in + c_0 + O\left(\frac{1}{n}\right) \) one obtains

\[
4\pi in \left( e^{\frac{3}{2}(a_1+a_2+2c_0)} - e^{-\frac{3}{2}(a_1+a_2+2c_0)} \right) + O(1) = 0
\]
and hence
\[c_0^{(s)} = -\frac{a_1 + a_2}{2} + \frac{s\pi i}{6}, \quad s \in \{0, \ldots, 5\}\]

from the equation \(\sinh\left(\frac{3}{2}\tilde{\lambda}_j(\lambda_n)\right) = 0\) we have
\[3a_j + 3c_0^{(s)} + O\left(\frac{1}{n}\right) = 2\pi is\]
\[c_0^s = -a_j + \frac{2\pi is}{3}, \quad s \in \{0, 1, 2\}\]

three sequences of eigenvalues \(-a_1, -a_2, -\frac{a_1 + a_2}{2}\)
spectrum of a graph in the previous figure, $a_1 = 2$, $a_2 = 1$, $b_1 = 0$, $b_2 = 0$
Example – One loop and an appendix

- Dirichlet coupling at the boundary vertex, standard coupling otherwise
- secular equation:

\[
\sin \left( \frac{3}{2} \tilde{\lambda}_2(\lambda) \right) \left( \tilde{\lambda}_1(\lambda) \cosh \frac{3\tilde{\lambda}_2(\lambda)}{2} \cosh (\tilde{\lambda}_1(\lambda)) + 
\right.
\]

\[
+ 2\tilde{\lambda}_2(\lambda) \cosh \frac{3\tilde{\lambda}_2(\lambda)}{2} \sinh (\tilde{\lambda}_1(\lambda)) = 0
\]
spectrum of a graph in the previous figure, $a_1 = 3$, $a_2 = 0$, $b_1 = 0$, $b_2 = 0$
more detailed look
more detailed look
Example – star graph with nonequal lengths of the edges

- spectrum of a star graph with different lengths of the edges,
  \( l_1 = 1, \ l_2 = 1, \ l_3 = 1.03 \)
Conclusion

- for an equilateral graph there is at most $2N$ high frequency abscissas
- for a bipartite equilateral graph there is at most $N$ high frequency abscissas
- for a tree equilateral graph with odd degree of vertices there exists such a damping that there is at least $N$ high frequency abscissas
P. Freitas, J. Lipovský: Eigenvalue asymptotics for the damped wave equation on metric graphs

Thank you for your attention!
Reference

P. Freitas, J. Lipovsky: Eigenvalue asymptotics for the damped wave equation on metric graphs

Thank you for your attention!