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Small-ε behavior of the non-Hermitian $\mathscr{PT}$-symmetric Hamiltonian $H = p^2 + x^2(ix)^\varepsilon$. (English summary)
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This paper contributes a long-missing piece of the puzzle to the branch of constructive quantum mechanics, also known as crypto-Hermitian, quasi-Hermitian, $\mathscr{PT}$-symmetric or pseudo-Hermitian quantum mechanics. In this approach the building of new tractable phenomenological models is based on their simplification via a nonstandard definition (i.e., a nontrivial, metric-mediated and, in general, strongly ambiguous redefinition) of the dual elements in the Hilbert space of states (i.e., in Dirac's terminology, of the so-called bra vectors—a more detailed version of this abstract explanation can be found in [F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Physics **213** (1992), no. 1, 74–101; MR1144600 (94a:81028)] or in [M. Znojil, SIGMA Symmetry Integrability Geom. Methods Appl. **5** (2009), Paper 001, 19 pp.; MR2470412 (2010e:81089)]). Concerning the particular, illustrative, anharmonic-oscillator example given in the title, some of its puzzling features had already been known for more than thirty years. The history (partly told to the reviewer by G. Alvarez; cf. also his comprehensive text in [J. Phys. A **28** (1995), no. 16, 4589–4598; MR1352176 (96g:81057)]) starts, perhaps, with the half-forgotten perturbation-theory paper on the asymptotically purely imaginary anharmonicity where, asymptotically, $\varepsilon = 1$ [cf. E. Caliceti, S. Graffi and M. Maioli, Comm. Math. Phys. **75** (1980), no. 1, 51–66; MR0581569 (82d:81030)]. Still, the paper under consideration seems to be the first successful attempt at introducing the above-mentioned definition via perturbation series using the smallness of the exponent ε. Psychologically, this is a slightly surprising fact since it was precisely Bender et al. who developed the appropriate method (called δ-expansions [cf. C. M. Bender et al., Phys. Rev. D (3) **37** (1988), no. 6, 1472–1484; MR0934829 (89g:81053); J. Math. Phys. **30** (1989), no. 7, 1447–1455; MR1002247 (90i:34085)]) five years earlier than their attention was re-directed to the apparently non-Hermitian Hamiltonians in question (by Daniel Bessis

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[private conversation with C. M. Bender, C. E. N. Saclay, 1993; per bibl.]).   *M. Znojil* (Řež u Prahy)