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Review text:

In an overlap (or union?) of the fields of mathematical physics, of spectral theory and of stochastic analysis (let us abbreviate it as PSS), the interest in non-self-adjoint operators is quickly growing during the recent years. Together with the authors, let me redirect all readers which would like to know the reasons to the Brian Davies' 2007 book cited as [8]. Adding, for the basic reference, also the citation of SPECTRA AND PSEUDOSPECTRA by Lloyd N. Trefethen and Mark Embree (Princeton University Press, Princeton and Oxford, 2005) with its "sixty short sections" such that "nobody will read them all", with readers still advised to "look at all the figures".

In the present review (prepared, originally, for the PSS conference in Goslar in 2011) there are no figures. The methods are taken, basically, from the authors' two previous works [27] and [25] on the distribution of eigenvalues of non-selfadjoint operators. The former approach (via complex analysis, CA) relies heavily on Borichev, Golinskii and Kupin as predecessors ([5], cf. Theorem 3.3.1). The eigenvalues are interpreted as zeros of a holomorphic function d_α called perturbation determinant. Under the accepted restrictions it is important that the (extended) resolvent set of the unperturbed operator is conformaly equivalent to the unit disk.

Without further specific assumptions the alternative, operator-theoretic approach using the concept of numerical range looks more appropriate for the underlying technical study of the rate of accumulation of the isolated eigenvalues towards a point in essential spectrum. The reason is that, due to assumptions, the closure of the numerical range and the spectrum of the unperturbed operator coincide. The key initial reference is made to several variants of the classical estimate (5.1.1) which is due to Kato.

In applications to concrete operators, each of the two approaches in question is shown to have specific advantages over the other. For the sake of definiteness, the choice of the operators is often restricted, typically, to the order$-p$ Schatten-class perturbations of self-adjoint bounded operators and/or to the relatively-Schatten perturbations of non-negative operators. In both approaches the eigenvalue moments are put in sums which are, typically, bounded in terms of the Schatten norm of the non-self-adjoint perturbation. In general, adding assumptions is usually observed to strengthen the relative merits of the initially less efficient CA approach.

Ultimately, in a way motivated by the role of the Lieb-Thirring inequalities (cf. (7.2.4) in self-adjoint case) in studies of the stability of quantum world [35], specific applications of the abstract results to certain non-self-adjoint Jacobi and Schrödinger operators are considered. Theorem 7.1.7 may be perceived as one of the paper's most typical and most relevant "applied-theory" results. It appears efficient, in particular, when a sequence of eigenvalues converges to an endpoint of the essential spectrum. The authors also add a list of most interesting open questions claiming, e.g., that the "validity or falsehood of estimate (7.2.12)" may be perceived to be "one of the major open problems" at present.

In a broader phenomenological context of growing popularity and of an active current use of non-self-adjoint Hamiltonians with real spectra in quantum physics (cf., e.g., the Mostafazadeh's extensive review paper "Pseudo-Hermitian Quantum Mechanics" in Int. J. Geom. Meth. Mod. Phys. 7 (2010) 1191-1306 (arXiv: 0810.5643)), I would like to add a remark that the turn of attention to the Lieb-Thirring inequalities in non-self-adjoint setup might in fact be also perceived as an ill-devised strategy. Typically, in the "most natural generalization of the self-adjoint Lieb-Thirring inequalities to the non-self-adjoint setting", Frank, Laptev, Lieb and Seiringer [16] were forced to consider just the *non-real* energy eigenvalues (*a "contradictio in adjecto"?*) in the sectors avoiding the positive half-line.