Weakly non-Hermitian square well

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Inside a box of size L we contemplate the simplest $\mathcal{PT}$-symmetric piece-wise constant potential of size $\ell < L$ and purely imaginary strength ig and describe all its bound states in closed form.

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Symmetric Dirichlet boundary conditions $\psi(\pm L) = 0$ make the one-dimensional Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\psi(x) \qquad (1)$$

exactly solvable in the Hermitian square-well case with $V(x) = 0$ and with the well-known quadratic spectrum

$$E_n = \left(\frac{n\pi}{2L}\right)^2, \qquad n = 1, 2, \ldots . \qquad (2)$$

A "minimal" modification of this model will be studied here, with $V(x)$ vanishing at $x \in (-L, -\ell)$ ("far left" region FL) and at $x \in (\ell, L)$ ("far right" region FR). This modification will be parity-pseudo-Hermitian (usually [1] called $\mathcal{PT}$-symmetric), i.e., we shall have $V(x) = -\mathrm{i}\,g$ at $x \in (-\ell, 0)$ (in the "near left" region NL) and $V(x) = \mathrm{i}\,g$ at $x \in (0, \ell)$ ("near right" region, NR). This means that at the sufficiently small ℓ or g the energies remain real, discrete and positive [2], with $E = k^2$, $k > 0$. In a way paralleling the older papers on similar models [3] this enables us to put

$$\psi_{FL}(x) = A^* \sin[k(L + x)], \qquad \psi_{FR}(x) = A \sin[k(L - x)]$$

and, with a complex $\kappa = s + \mathrm{i}t$ such that $s \geq 0$, $t \geq 0$ and $g = 2st$,

$$\psi_{NL}(x) = B^* \cosh(\kappa^* x) + \mathrm{i}D^* \sinh(\kappa^* x) \equiv \psi^*_{NR}(-x)\,.$$

As long as $k^2 = t^2 - s^2$ and B (as well as $C = \kappa D$) must be all real, the matching conditions

$$\psi_{FR}(\ell) = \psi_{NR}(\ell) \qquad \partial_x \psi_{FR}(\ell) = \partial_x \psi_{NR}(\ell)$$

yield finally the two complex linear algebraic equations for the four unknown real constants B, C, $\mathrm{Re}\, A$ and $\mathrm{Im}\, A$. Of course, their secular determinant must vanish,

$$k \sin[2k(L-\ell)][s^2 \cosh(2s\ell) + t^2 \cos(2t\ell)] - \cos[2k(L-\ell)][s^3 \sinh(2s\ell) - t^3 \sin(2t\ell)]$$
$$+st^2 \sinh(2s\ell) - s^2 t \sin(2t\ell) = 0. \qquad (3)$$

In the regime of the weak-non-Hermiticity (and, say, in units $L = 1$ and under above-mentioned constraint $g = 2st$) all the roots of this equation are real [2] and determine all the energies as functions of g and ℓ. On the boundary of the related two-dimensional domain of parameters (g, ℓ) the roots as well as the related pairs of neighboring energies $E_{m_{1,2}}$ merge and become complex at the so called critical or exceptional pairs (g_c, ℓ_c) as sampled in Table 1.

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Table 1. Critical strength g_c vs. critical range ℓ_c. Renormalized quantities $\sqrt[12]{g_c}$ are fitted by the asymptotially logarithmic function $F(\ell_c) = \sqrt[2]{(\ln(1/\ell_c))^2/6 + \alpha}$ chosen as exact at $\ell = L = 1$ and, apparently, offering an upper bound at the smallest ℓ.

ℓ_c	1.00	0.70	0.50	0.40	0.30	0.20	0.10	0.01	0.001
g_c	4.475	4.813	6.436	8.601	13.43	27.27	95.83	9895.	486950.
$F(\ell_c)$	1.133	1.142	1.168	1.193	1.235	1.310	1.472	2.195	3.039
$\sqrt[12]{g_c}$	1.133	1.140	1.168	1.196	1.242	1.317	1.463	2.153	2.978