News in non-Hermitian Quantum Mechanics

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PLAN OF THE TALK

- **I.** Introduction
- **II.** models on curves
- **III.** models on an interval
- **IV.** summary

CONTENTS OF THE TALK

- **I.** Introduction **(2 $\times$ 2 example)**
- **II.** models on curves **("toboggans")**
- **III.** models on an interval **(coupled square wells)**
- **IV.** summary

in the other words:

- **I.** motivation, so called PT symmetry etc
- **II.** single channel: non-Hermitian harmonic oscillator etc.
- **III.** non-Hermitian coupling of channels: Klein Gordon etc.
- **IV.** briefly: an innovation of quantization recipes.

sleepers: **partly accessible ON WEB** and published:

- **I.** quant-ph/0601048, Phys. Lett. A 353 (2006) 463 - 468 (generalized PT symmetry).
- **II.** quant-ph/0502041, Phys. Lett. A 342 (2005) 36 - 47 and quant-ph/0606166, J. Phys. A: Math. Gen. 39 (2006) 13325 - 13336 (oscillations along complex curves).

- **III.** quant-ph/0511194, J. Phys. A: Math. Gen. 39 (2006) 4047 - 4061 (coupled square wells) and quant-ph/0605209, J. Phys. A: Math. Gen. 39 (2006) 10247 - 10261 (discrete version: Runge Kutta oscillator).
- **IV.** Czechosl. J. Phys. 56 (2006) 977 - 984 (a concise review).

Prelude

we wish that

$$H^\dagger = \mathbf{R}\, H\, \mathbf{R}^{-1}$$

observation 1

if we wish that

$$H^{\dagger} = \mathbf{R}\, H\, \mathbf{R}^{-1}$$

then

$$H = \left(H^{\dagger}\right)^{\dagger} = \left(\mathbf{R}^{-1}\right)^{\dagger} \quad H^{\dagger} \quad \mathbf{R}^{\dagger}$$

observation 2

if we wish that

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then

$$H = \left(H^\dagger\right)^\dagger = \left(\mathbf{R}^{-1}\right)^\dagger \quad H^\dagger \quad \mathbf{R}^\dagger$$

$$H = \left(H^\dagger\right)^\dagger = \left(\mathbf{R}^{-1}\right)^\dagger \mathbf{R}\, H\, \mathbf{R}^{-1}\mathbf{R}^\dagger$$

observation 3

if we wish that

$$H^{\dagger} = \mathbf{R}\, H\, \mathbf{R}^{-1}$$

then

$$H = \left(H^{\dagger}\right)^{\dagger} = \left(\mathbf{R}^{-1}\right)^{\dagger} \quad H^{\dagger} \quad \mathbf{R}^{\dagger}$$

$$H = \left(H^{\dagger}\right)^{\dagger} = \left(\mathbf{R}^{-1}\right)^{\dagger} \mathbf{R}\, H\, \mathbf{R}^{-1} \mathbf{R}^{\dagger}$$

$$H = \left(H^{\dagger}\right)^{\dagger} = \mathcal{S} \quad H \quad \mathcal{S}^{-1}$$

This means that

our H must have a **factorized** symmetry,

$$H\,\mathcal{S} = \mathcal{S}\,H, \qquad \mathcal{S} = \left(\mathbf{R}^{-1}\right)^{\dagger}\mathbf{R}$$

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and we have two possibilities:

- **either** $\mathcal{S} = I$ (i.e., $\mathbf{R} = \mathbf{R}^{\dagger}$), *pseudo-Hermiticity*

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and we have two possibilities:

- **either** $\mathcal{S} = I$ (i.e., $\mathbf{R} = \mathbf{R}^{\dagger}$), *pseudo-Hermiticity*
- **or** $\mathcal{S} \neq I$ (*"symmetry factorization"*, SF).

INTRODUCTION

With $H|n\rangle = E_n|n\rangle$ and $\langle\langle n|H = E_n\langle\langle n|$, quasi-Hermiticity

$$H^\dagger = \Theta\, H\, \Theta^{-1}, \qquad I \neq \Theta = \Theta^\dagger > 0.$$

and the spectral representation of the Hamiltonian

$$H = \sum_n |n\rangle \frac{E_n}{\langle\langle n|n\rangle} \langle\langle n|$$

this leads to the multiparametric formula giving **"physics"**,

$$\Theta = \sum_n |n\rangle\rangle\, \theta_n \,\langle\langle n|\,, \qquad\qquad \theta_n > 0.$$

Example – find metric Θ for a 2×2 Hamiltonian

$$H = \begin{pmatrix} -T & B \\ \\ -B & T \end{pmatrix}, \qquad \Theta = \begin{pmatrix} a & b \\ \\ b & d \end{pmatrix}$$

$$\Theta H = H^T \Theta \Longrightarrow 2bT = -B(a+d)$$

$$E \in I\!R \Longleftrightarrow |T| \geq |B|, \quad B = T \, \sin \alpha$$

$$\theta_{1,2} > 0 \Longleftrightarrow b \neq 0 \neq a + d = 2Z.$$

ambiguity:

for $a = Z(1+\xi)$, $d = Z(1-\xi)$ we have an **interval**,

$$1 > \sqrt{\xi^2 + \sin^2\alpha}, \qquad \xi < \cos\alpha.$$

interpretation:

In 2D with biorthogonal "brabraket" basis,

$$\langle\langle n|\, H = \langle\langle n|\, E_n, \qquad H\,|n\rangle = E_n\,|n\rangle$$

ambiguity is compatible with the universal formula

$$\Theta = \boldsymbol{\Sigma}\ |n\rangle\rangle\, s_n\, \langle\langle n|\,, \qquad s_k > 0\,.$$

MODELS ON COMPLEX CONTOURS $\mathcal{C}^{(N)}$

FIRST STEP: SPIKED HO

Miloslav Znojil,

PT symmetric harmonic oscillators

Phys. Lett. A 259 (1999) 220 - 3.

Innovation: PT-symmetric paths $\mathcal{C}^{(N)}$ N-times encircle $x = 0$,

$$\left(-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + x^2\right) \psi(x) = E\,\psi(x)$$

to be studied in the bound-state and scattering regime

1. along straight contour

$$\mathcal{C}^{(0)} = \{x \mid x = t - i\,\varepsilon,\, t \in I\!\!R\}$$

"twice as many" bound-state levels

$$E = E_{n,\ell,\pm} = 4n + 2 \pm 2\alpha(\ell)$$

2. along loops

$$\mathcal{C}^{(N)} = \mathcal{D}_{(\varepsilon,N)}^{(PTSQM,\,tobogganic)}$$

new quantum theory:

on multisheeted Riemann surfaces

with, say, $\varphi \in (-(N+1)\pi,\ N\pi)$ in

$$\varrho(\varphi, N) = \sqrt{1 + \tan^2 \frac{\varphi + \pi/2}{2N+1}}$$

$$\mathcal{C}^{(N)} = \left\{ x = \varepsilon\, \varrho(\varphi, N)\, e^{i\,\varphi}\,,\, \varepsilon > 0 \right\}\ .$$

What is $\mathcal{PT}$−symmetry in the presence of branch points?

rotation along Riemann surface.

SECOND STEP: AHOs in QES regime

Miloslav Znojil (quant-ph/0502041):

PT-symmetric quantum toboggans

Phys. Lett. A 342 (2005) 36-47.

$$\left[-\frac{d^2}{dx^2} + V(x)\right] \psi(x) = E\,\psi(x)$$

$$\mathrm{Re}\,V(x) = +\mathrm{Re}\,V(-x) \text{ and } \mathrm{Im}\,V(x) = -\mathrm{Im}\,V(-x).$$

$$\psi(\pm\mathrm{Re}\ L + i\,\mathrm{Im}\ L) = 0\,, \qquad |L| \gg 1 \quad \text{or} \quad |L| \to \infty\,.$$

model:

$$V(x) = x^{10} + \text{asymptotically smaller terms}$$

$$\psi(x) = e^{-x^6/6 + \text{asymptotically smaller terms}}$$

reparametrized

$$\psi(x) = \exp\left[-\frac{1}{6}\varrho^6 \cos 6\varphi + \text{asymptotically less relevant terms}\right] ,$$

closed formulae

$$\Omega_{(first\ right)} = \left(-\frac{\pi}{2} + \frac{\pi}{12}, -\frac{\pi}{2} + \frac{3\pi}{12}\right),$$

$$\Omega_{(first\ left)} = \left(-\frac{\pi}{2} - \frac{\pi}{12}, -\frac{\pi}{2} - \frac{3\pi}{12}\right),$$

$$\Omega_{(third\ right)} = \left(-\frac{\pi}{2} + \frac{5\pi}{12}, -\frac{\pi}{2} + \frac{7\pi}{12}\right), \quad \ldots$$

$$\ldots \quad \Omega_{(fifth\ left)} = \left(-\frac{\pi}{2} - \frac{9\pi}{12}, -\frac{\pi}{2} - \frac{11\pi}{12}\right).$$

$\mathcal{PT}-$**symmetric transformations changing** β

Initial $\mathcal{PT}-$symmetric model

$$\left[-\frac{d^2}{dx^2} - (ix)^2 + \lambda\, W(ix)\right]\,\psi(x) = E(\lambda)\,\psi(x)\,,$$

sample potential:

$$W(ix) = \sum_\beta g_\beta (ix)^\beta \,.$$

change variables,

$$ix = (iy)^\alpha \,, \qquad \psi(x) = y^\varrho \, \varphi(y).$$

in detail:

at $\alpha > 0$ we have

$$i\,dx = i^{\alpha}\alpha y^{\alpha-1}\,dy, \qquad \frac{(iy)^{1-\alpha}}{\alpha}\,\frac{d}{dy} = \frac{d}{dx}\,.$$

"new" Schrödinger equation

looking complicated:

$$y^{1-\alpha}\frac{d}{dy}y^{1-\alpha}\frac{d}{dy}\,y^{\varrho}\,\varphi(y)+i^{2\alpha}\alpha^{2}\left[-(iy)^{2\alpha}+\lambda\,W[(iy)^{\alpha}]-\right.$$

$$\left.-E(\lambda)\right]\,y^{\varrho}\,\varphi(y)=0\,.$$

first term:

$$y^{1-\alpha}\frac{d}{dy}y^{1-\alpha}\frac{d}{dy}\,y^{[(\alpha-1)/2]}\,\varphi(y) =$$

$$= y^{2+\varrho-2\alpha}\frac{d^2}{dy^2}\,\varphi(y) + \varrho(\varrho-\alpha)y^{\varrho-2\alpha}\,\varphi(y)\,, \quad \varrho = \frac{\alpha-1}{2}$$

Thus, the new Schrödinger equation is

of the same form:

$$-\frac{d^2}{dy^2}\,\varphi(y) + \frac{\alpha^2-1}{4y^2}\,\varphi(y) + (iy)^{2\alpha-2}\alpha^2\left[-(iy)^{2\alpha} + \lambda\,W[(iy)^{\alpha}]\right.$$

$$\left.-E(\lambda)\right]\,\varphi(y) = 0\,.$$

Example: Quasi-exact toboggans:

$$V_f(x) = x^6 + f_4\, x^4 + f_2\, x^2 + f_{-2}\, x^{-2},$$

$$V_g(y) = -(iy)^2 + i\, g_1\, y + g_{-1}\, (iy)^{-1} + g_{-2}\, (iy)^{-2},$$

$$V_h(y) = -(iy)^{2/3} + h_{-2/3}\, (iy)^{-2/3} + h_{-4/3}\, (iy)^{-4/3} + h_{-2}\, (iy)^{-2}$$

mutually interrelated, with $\alpha = 1$ for V_f and $\alpha = 1/2$ for V_g or $\alpha = 1/3$ for V_h.

Summary: Perturbed harmonic oscillator

$$V(x) = x^2 + \sum_{\beta} g_{(\beta)}\, x^{\beta} \tag{1}$$

lives on **topologically nontrivial trajectories** $\mathcal{C}^{(N)}$. Its

$$\psi(x) \approx \psi^{(\pm)}(x) = e^{\pm x^2/2}\,, \qquad |x| \gg 1 \tag{2}$$

= multivalued analytic functions. A ray $x_\theta = \varrho e^{i\,\theta}$ chosen.

“physical” [i.e., asymptotically vanishing $\psi^{(phys)}(x)$] and

“unphysical” [i.e., asymptotically “exploding” $\psi^{(unphys)}(x)$],

$$\psi^{(-)}(x) = \left\{ \begin{array}{ll} \psi^{(phys)}(x), & k\pi + \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right), \\ & \\ \psi^{(unphys)}(x), & k\pi + \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \end{array} \right. , \qquad k \in \mathbb{Z}$$

alternatively,

$$\psi^{(+)}(x) = \begin{cases} \psi^{(unphys)}(x), & k\pi + \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right), \\ \psi^{(phys)}(x), & k\pi + \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \end{cases}, \qquad k \in \mathbb{Z} \, .$$

Riemann-surface values of the "tobogganic trajectories"

$$\mathcal{D}_{(\varepsilon,N)}^{(PTSQM,\,tobogganic)} = \left\{ x = \varepsilon\, \varrho(\varphi, N)\, e^{i\,\varphi} \mid \varphi \in (-(N+1)\pi,\ N\pi) \right\}$$

$$\varrho(\varphi, N) = \sqrt{1 + \tan^2 \frac{\varphi + \pi/2}{2N+1}}$$

Bound states

$$H_{(\mathcal{PT})}\,\psi(x) = E\,\psi(x)$$

with Dirichlet inside the wedges,

$$\psi\left(\varrho\cdot e^{i\,\theta}\right) = 0, \qquad \varrho \gg 1 \qquad \theta + k_{i,f}\,\pi \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

defines bound states, often with real spectra.

For toboggans we selected $k_f = 0$ and $k_i = 1$ at $N = 0$,

$k_f = -1$ and $k_i = 2$ at $N = 1$,

$k_f = -2$ and $k_i = 3$ at $N = 2$ etc.

Scattering along the toboggans

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independent solutions become equally large and oscillate not only for $E > 0$ when $V(x) = 0$ at $\varrho \to \infty$ but also for any potential including our x^2-dominated one.

"in" and "out" wedge boundaries are

$$\mathcal{A}^{(N)}_{(L)} \to \varrho\, e^{i\,\theta_{in}}, \qquad \theta_{in} = -(N+3/4)\,\pi,$$

$$\mathcal{A}^{(N)}_{(L)} \to \varrho\, e^{i\,\theta_{out}}, \qquad \theta_{out} = (N-1/4)\,\pi,$$

$$\mathcal{A}^{(N)}_{(U)} \to \varrho\, e^{i\,\theta_{in}}, \qquad \theta_{in} = -(N+5/4)\,\pi,$$

$$\mathcal{A}^{(N)}_{(U)} \to \varrho\, e^{i\,\theta_{out}}, \qquad \theta_{out} = (N+1/4)\,\pi\,.$$

We require the following incoming-beam normalization,

$$\psi\left(\varrho \cdot e^{i\,\theta_{in}}\right) = \psi_{(i)}(x) + B\,\psi_{(r)}(x), \qquad \varrho \gg 1, \quad \theta_{in} = \text{fixed}$$

and outcoming-beam normalization,

$$\psi\left(\varrho \cdot e^{i\,\theta_{out}}\right) = (1+F)\,\psi_{(t)}(x), \qquad \varrho \gg 1, \quad \theta_{out} = \text{fixed}$$

with incident and reflected waves $\psi_{(i,r)}(x) \approx e^{\pm i\varrho^2/2}$.

$B =$ "backward scattering" and $F =$ "forward scattering"

Exactly solvable model of scattering on x^2

$$\left[-\frac{d^2}{dx^2} + \frac{\alpha^2 - 1/4}{x^2} + x^2\right] \psi(x) = E\,\psi(x), \qquad \alpha = \ell + \frac{1}{2},$$

set $x^2 = -ir$ along the first nontrivial scattering path $\mathcal{A}^{(0)}_{(L)}$.

“in” branch with $r \ll -1$ and “out” branch with $r \gg +1$

$$\chi_{(\alpha)}(r) = r^{\frac{1}{4}+\frac{\alpha}{2}}\, e^{ir/2}\, {}_1F_1\left(\frac{\alpha+1-\mu}{2}, \alpha+1; -ir\right), \qquad E = 2\mu$$

linearly independent partner $\chi_{(-\alpha)}(r)$ ($\alpha \neq n \in I\!N$).

$|r| \gg 1$ estimate,

$$r^{\frac{1}{4}+\frac{\alpha}{2}}\chi_{(\alpha)}(r) \approx \; e^{ir/2}\,\frac{r^{\mu/2}\;\exp\left[-i\,\pi\;(\alpha+1)\,/4\right]}{\Gamma\left[(\alpha+1+\mu)/2\right]}+$$

$$+e^{-ir/2}\,\frac{r^{-\mu/2}\;\exp\left[+i\,\pi\;(\alpha+1)\,/4\right]}{\Gamma\left[(\alpha+1-\mu)/2\right]}\,.$$

"rigid" at $\alpha > 0$, $\mu = E/2 > 0$ and $|x| = |\sqrt{(}r)| \gg 1$

$$\psi_{in,out}(x) \approx r^{-1/4+(\alpha+\mu)/2}\, e^{ir/2}\, \frac{\exp\left[-i\,\pi\ \left(-\alpha+1\right)/4\right]}{\Gamma\left[(-\alpha+1+\mu)/2\right]} + \dots .$$

Note that $\psi_{out}^{(Coul)}(r)$ becomes "distorted" by power-law as well,

$$\sin(\kappa r + const) \to \sin(\kappa r + const \cdot \log r + const)\,.$$

Toboggans in potentials with more spikes

two branch points (say, in $x = \pm 1$)

$$V(x) = x^2 + \frac{G}{(x-1)^2} + \frac{G^*}{(x+1)^2}$$

Sub-summary of the tobogganic study

Quantum particle is assumed moving along $\mathcal{PT}-$symmetric "toboggan" paths which $N-$times encircle the branch point in the origin. Both bound states and scattering.

MODELS ON AN INTERVAL

FIRST STEP: K COUPLED SQUARE WELLS

RECOLLECT our main idea: work with **non-metric**,

$$\mathbf{P} \neq \mathbf{P}^\dagger$$

pattern: if $H^\dagger = \mathbf{R}\, H\, \mathbf{R}^{-1}$ and $\mathbf{R} \neq \mathbf{R}^\dagger$,

we have the symmetry,

$$H\, \mathcal{S} = \mathcal{S}\, H, \qquad \mathcal{S} = [\mathbf{R}^{-1}]^\dagger \mathbf{R}.$$

Let's choose $\mathbf{R}^{-1} = \mathbf{R}^{\dagger}$ with $\mathcal{S} = \mathbf{R}^{2}$ and

$$\mathbf{R} = \begin{pmatrix} 0 & \dots & 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & \dots & 0 & 0 \\ 0 & \mathcal{P} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \mathcal{P} & 0 \end{pmatrix}$$

at any K.

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Toy model with two coupled channels

(a) Hamiltonian:

$$H_{(kinetic)} = \begin{pmatrix} -\frac{d^2}{dx^2} & 0 \\ & \\ 0 & -\frac{d^2}{dx^2} \end{pmatrix},$$

$$H_{(interaction)} = \begin{pmatrix} V_a(x) & W_b(x) \\ & \\ W_a(x) & V_b(x) \end{pmatrix}.$$

(b) its $\theta-$pseudo-Hermiticity:

$$\theta = \theta^{\dagger} = \begin{pmatrix} 0 & \mathcal{P} \\ \mathcal{P} & 0 \end{pmatrix} = \theta^{-1}$$

(c) potentials $[x \in (-1,0)]$:

$$\operatorname{Im} W_a(x) = X > 0,$$

$$\operatorname{Im} W_b(x) = Y > 0,$$

$$\operatorname{Im} V_a(x) = \operatorname{Im} V_b(x) = Z,$$

(d) spin-like ($\sigma = \pm 1$) symmetry:

$$\Omega = \begin{pmatrix} 0 & \omega^{-1} \\ \\ \omega & 0 \end{pmatrix}, \qquad \omega = \sqrt{\frac{X}{Y}} > 0.$$

(e) solvable and physical

(f) simple in a modified Dirac's notation

$$H|E,\sigma\rangle = E|E,\sigma\rangle, \quad \Omega\,|E,\sigma\rangle = \sigma\,|E,\sigma\rangle$$

$$\langle\!\langle E,\sigma|H = E\langle\!\langle E,\sigma|, \quad \langle\!\langle E,\sigma|\Omega = \sigma\langle\!\langle E,\sigma|$$

$$\text{biog.}: \quad 0 = \langle\!\langle E', \sigma' | E, \sigma \rangle \times \begin{cases} (E' - E) \\ \\ (\sigma' - \sigma) \end{cases}$$

$$\text{cpl.}: I = \sum_{E,\sigma} \; |E, \sigma\rangle \, \frac{1}{\langle\!\langle E, \sigma | E, \sigma \rangle} \; \langle\!\langle E, \sigma|$$

$$\text{sp.}: \; H = \sum_{E,\sigma} \; |E, \sigma\rangle \, \frac{E}{\langle\!\langle E, \sigma | E, \sigma \rangle} \langle\!\langle E, \sigma|$$

$$\Omega = \sum_{E,\sigma} \; |E, \sigma\rangle \, \frac{\sigma}{\langle\!\langle E, \sigma | E, \sigma \rangle} \langle\!\langle E, \sigma|$$

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FULL MODEL WITH K COUPLED SQUARE WELLS

$$V(x) = V_{(Z)}(x) = -\mathrm{i}\, Z\, \mathrm{sign}(x), \qquad x \in (-1, 1)$$

STILL HAS ITS MERITS!

(a) = **ODE** with constant coefficients:

$$-\frac{d^2}{dx^2}\,\varphi^{(m)}(x) + \sum_{j=1}^{K} V_{Z_{(m,j)}}(x)\,\varphi^{(j)}(x) =$$

$$= E\varphi^{(m)}(x), \qquad m = 1, 2, \ldots, K$$

(b) = **solvable** by an ansatz for $\varphi^{(m)}(x)$

$$= \begin{cases} C_L^{(m)} \sin \kappa_L(x+1), & x < 0, \\ C_R^{(m)} \sin \kappa_R(-x+1), & x > 0 \end{cases}$$

(c) = giving $Z^{(m)}_{(eff)}(K)$ as eigenvalues of

$$\begin{pmatrix} Z_{(1,1)} & Z_{(1,2)} & \dots & Z_{(1,K)} \\ Z_{(2,1)} & Z_{(2,2)} & \dots & Z_{(2,K)} \\ \vdots & \ddots & \ddots & \vdots \\ Z_{(K,1)} & Z_{(K,2)} & \dots & Z_{(K,K)} \end{pmatrix}.$$

(d) quantized easily:

$= \text{ansatz} \to \kappa_R = s + \mathrm{i}t = \kappa_L^*, \quad s > 0,$

$\to t = t_{first\ curve}(s) = Z^{(m)}_{(eff)}(K)/(2s)$

plus **matching** in the origin: PTO

$\rightarrow \kappa_L \,\mathrm{cotan}\, \kappa_L = -\kappa_R \,\mathrm{cotan}\, \kappa_R$

gives the second, "universal" curve

$t = t_{exact}(s)$ with implicit definition

$$\boxed{2s\ \sin 2s + 2t\ \sinh 2t = 0}$$

$\rightarrow$ **energies** via **intersections** at any K,

$$E_n = s_n^2 - t_n^2, \qquad n = 0, 1, \ldots\ .$$

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Technicalities

(a) take generalized parities $\mathbf{R} = \mathbf{R}_{(K,L)} = \mathcal{P}\,\mathbf{r}_{(K,L)}$,

$$\mathbf{r}_{(K,L+1)} = \mathbf{r}_{(K,1)}\,\mathbf{r}_{(K,L)}\,, \quad L = 1, 2, \ldots\,.$$

$\left[\mathbf{r}_{(K,L)}\right]^K = I$, $\mathbf{r}_{(K,K-L)} = \left[\mathbf{r}_{(K,L)}\right]^\dagger$.

(b) adapt H to $\mathbf{R}$:

$$\mathbf{A} = \mathbf{r}_{(K,K-L)} \cdot \mathbf{A}^T \cdot \mathbf{r}_{(K,L)}$$

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Let us pick up

THREE channels

$$\mathbf{R}_{(3,1)} = \begin{pmatrix} 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & 0 \\ 0 & \mathcal{P} & 0 \end{pmatrix} = \mathbf{R}^{\dagger}_{(3,2)} = \mathbf{R}^{-1}_{(3,2)},$$

$$\mathbf{R}_{(3,2)} = \begin{pmatrix} 0 & \mathcal{P} & 0 \\ 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & 0 \end{pmatrix} = \mathbf{R}^{\dagger}_{(3,1)} = \mathbf{R}^{-1}_{(3,1)}.$$

giving the **unique**

$$\mathbf{A}_{(interaction)} = \begin{pmatrix} Z & X & X \\ X & Z & X \\ X & X & Z \end{pmatrix}, \quad L = 1, 2$$

the 'first curve' $t = t^{(\sigma)}(s) = \frac{1}{2s}\, Z_{eff}(\sigma)$, $\sigma = 1, 2, 3$

$$Z_{eff}(1) = Z + 2\,X, \qquad Z_{eff}(2,3) = Z - X$$

$$\left(C^{(a)}_{(1)}, C^{(b)}_{(1)}, C^{(c)}_{(1)}\right) \sim (1, 1, 1)$$

$$\left(C^{(a)}_{(2)}, C^{(b)}_{(2)}, C^{(c)}_{(2)}\right) \sim (1, -1, 0)$$

$$\left(C^{(a)}_{(3)}, C^{(b)}_{(3)}, C^{(c)}_{(3)}\right) \sim (1, 1, -2)\,.$$

Energies real: $Y - Z_{crit} \leq Z \leq Z_{crit} - 2Y$.

[vertices $(0, \pm 4.475)$ and $(2.98, -1.49)$].

Numerical interlude

(a) weakly non-Hermitian regime:

$$s = s_n = \frac{(n+1)\pi}{2} + \tau \, \frac{Q_n}{2}, \quad \tau = (-1)^n$$

$\longrightarrow$ solvable by **iterations**:

the first small quantity $\varrho \equiv \frac{1}{L} = \frac{1}{(n+1)\pi}$

the second one $\alpha = \frac{2\, Z_{eff}(\sigma)}{L}$ or $\beta = \alpha \varrho$

$\longrightarrow$ a “generalized continued fraction”

$$Q = \arcsin\left(2t\,\frac{\varrho}{1+\tau\,Q\,\varrho}\,\sinh 2t\right), \quad 2t = \frac{\alpha}{1+\tau\,Q\,\varrho}.$$

(b) intermediate non-Hermiticities: *ad hoc*:

$\rightarrow \arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \ldots$

$Q = Q(\alpha, \beta) = \alpha\beta\,\Omega(\alpha, \beta),$

$\rightarrow \Omega(\alpha, \beta) = 1 + c_{10}\,\alpha^2 + c_{01}\,\beta^2 +$

$+ c_{20}\,\alpha^4 + c_{11}\,\alpha^2\beta^2 + c_{02}\,\beta^4 + \mathcal{O}(\alpha^6)$

→ equation **re-arranged**:

$$[1 + \tau\, \beta^2 \Omega(\alpha, \beta)]\, \mathrm{arcsinh}(\Lambda) = \alpha$$

$\Lambda = [1 + \tau\, \beta^2 \Omega(\alpha, \beta)]^2\, \frac{1}{\beta}\, \sin[\alpha\beta\, \Omega(\alpha, \beta)]$

(c) formulae:

$\rightarrow$ leading order relation

$$0 = \left(-\tfrac{1}{6} + c_{10} + c_{01}\varrho^2 + 3\tau\varrho^2\right)\alpha^3 + \ldots .$$

determines the first two coefficients,

$$c_{10} = \tfrac{1}{6}, \qquad c_{01} = -3\tau\,,$$

the next-order $O\left(\alpha^5\right)$ gives

$c_{20} = \frac{1}{120}, \qquad c_{11} = \frac{1-8\tau}{6}, \qquad c_{02} = 15$

and the $1 + O\left(\alpha^4\right)$ formula

$$Q_n = \frac{4\,Z_{eff}^2}{(n+1)^3\pi^3} +$$
$$+ \frac{8\,Z_{eff}^4}{3\,(n+1)^5\pi^5}\left(1 + \frac{18\,(-1)^{n+1}}{(n+1)^2\pi^2}\right).$$

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FOUR channels

$K = 4$ warning: $\mathbf{R}_{(4,2)}$ is Hermitian,

mere six constraints upon 16 couplings.

Not enough symmetry for us.

Unique coupling-matrix left,

$$\mathbf{A}_{(interaction)} = \begin{pmatrix} Z & U & D & U \\ L & Z & L & D \\ D & U & Z & U \\ L & D & L & Z \end{pmatrix}, \quad L = 1, 3.$$

solution :

Four shifts of the effective Z,

$[-D, -D, D + 2\sqrt{UL}, D - 2\sqrt{UL}]$

with respective eigenvectors

$$\{1, 0, -1, 0\}, \{0, 1, 0, -1\},$$

$$\left\{U, \pm\sqrt{UL}, U, \pm\sqrt{UL}\right\}.$$

remark:

from the pseudo-parity

$$\mathbf{r}^{(permuted)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

repartitioned model

$$\mathbf{A}^{(permuted)}_{(interaction)} = \left(\begin{array}{cc|cc} Z & D & U & U \\ D & Z & U & U \\ \hline L & L & Z & D \\ L & L & D & Z \end{array}\right), \quad L = 1, 3.$$

.

FIVE channels

$$\mathbf{r}_{(5,1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \ldots,$$

all lead to the same

$$\mathbf{A}_{(interaction)} = \begin{pmatrix} Z & X & D & D & X \\ X & Z & X & D & D \\ D & X & Z & X & D \\ D & D & X & Z & X \\ X & D & D & X & Z \end{pmatrix}.$$

$\rightarrow$ exceptional eigenvalue $F_0 = 2\,D + 2\,X$ giving eigenvector

$\{1, 1, 1, 1, 1\}$

$\rightarrow$ the reduced $Z = 0$ matrix A has the pair of the twice degenerate eigenvalues with 2 respective eigenvectors

$$F_{\pm} = \frac{1}{2}\left[-D - X \pm \sqrt{5}\,(-D + X)\right]$$

$$\left\{1 \mp \sqrt{5}, -1 \pm \sqrt{5}, 2, 0, -2\right\}$$

$$\left\{1 \mp \sqrt{5}, -2, 0, 2, -1 \pm \sqrt{5}\right\}.$$

.

When do the energies remain real?

(a) a numerical **algorithm**:

$$\left.\frac{Q}{2}\right|_{crit} \equiv \varepsilon(t_{crit}) = \pi - \frac{Z_{crit}}{2t_{crit}},$$

$$\sin\left[2\,\varepsilon(t)\right] = \frac{t\ \sinh\ 2t}{\pi - \varepsilon(t)},$$

$\varepsilon_{(lower)}(t) = \pi/4$ and $\varepsilon_{(upper)}(t) = 0$.

$$\partial_t \varepsilon(t_{crit}) = \frac{Z_{crit}}{2t_{crit}^2},$$

$$\partial_t \varepsilon(t) = \frac{\sinh\ 2t + 2t\ \cosh\ 2t}{2\ [\pi - \varepsilon(t)]\ \cos 2\varepsilon(t) - \sin 2\varepsilon(t)}$$

a sample result:

$\longrightarrow t_{crit} \in (0.839393459, 0.839393461)$,

$\longrightarrow s_{crit} \in (2.665799044, 2.665799069)$,

$\longrightarrow E_{crit} \in (6.401903165, 6.401903294)$.

Table 1:

iteration	$Z_{crit}^{(lower)}$	$Z_{crit}^{(upper)}$
0	4.299	4.663
2	4.4614	4.4857
4	4.47431	4.47601
6	4.475239	4.475357
8	4.47530381	4.4753119
10	4.475308262	4.475308823
12	4.475308560	4.475308614

.

SIX channels

$L = 3$

21 free parameters

Hermitian R and a weak symmetry,

skipped

$L = 1$ or $L = 5$:

$$\mathbf{r}_{(6,1)}^{(permuted)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$\mathbf{A}$ = asymmetric:

$$\mathbf{A}^{(permuted)}_{(interaction)} = \left(\begin{array}{cc|cc|cc} Z & Y & G & B & F & B \\ X & Z & C & F & C & G \\ \hline F & B & Z & Y & G & B \\ C & G & X & Z & C & F \\ \hline G & B & F & B & Z & Y \\ C & F & C & G & X & Z \end{array}\right).$$

eigenvalues = roots of quadratic equations

two = non-degenerate, two = doubly degenerate

$L = 2$ or $L = 4$:

$$\mathbf{r}_{(6,2)}^{(permuted)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$\mathbf{A}$ = symmetric:

$$\mathbf{A}^{(permuted)}_{(interaction)} = \left(\begin{array}{ccc|ccc} Z & X & X & C & D & G \\ X & Z & X & G & C & D \\ X & X & Z & D & G & C \\ \hline C & G & D & A & B & B \\ D & C & G & B & A & B \\ G & D & C & B & B & A \end{array}\right).$$

eigenvalues = roots of quadratic equations

two = non-degenerate, two = doubly degenerate

.

$2M-1$ channels with $M=4$ etc

$M = 4$: Cardano formulae, four parameters at all L:

$$\mathbf{A}_{(interaction)} = \begin{pmatrix} Z & X & Y & D & D & Y & X \\ X & Z & X & Y & D & D & Y \\ Y & X & Z & X & Y & D & D \\ D & Y & X & Z & X & Y & D \\ D & D & Y & X & Z & X & Y \\ Y & D & D & Y & X & Z & X \\ X & Y & D & D & Y & X & Z \end{pmatrix}.$$

.

$2M$ channels with $M = 4$ etc

37 free parameters for $(2M, L) = (8, 4)$ (29 in pairs),

16 free parameters for $(2M, L) = (8, 2)$ (all in quadruplets),

8 free parameters for $(2M, L) = (8, 1)$ etc, (all in octuplets).

Summarizing asymmetrically coupled square wells

(1) Recipe $\mathcal{P} \to \mathcal{R}$ allowing **finite rotations** = feasible.

(2) Models carrying **new type of symmetries**.

(3) **New** "quantum practice", **quasi-Hermitian**.

SECOND STEP: SQUARE WELLS

DISCRETIZED,

so that the REALITY OF SPECTRA

can be proved MORE easily,

by the standard MATRIX TECHNIQUES

Runge-Kutta recipe

$$x_0 = -1, \qquad x_k = x_{k-1} + h = -1 + kh,$$

$$h = \frac{2}{N}, \qquad k = 1, 2, \ldots, N$$

$$-\psi''(x) \approx -\frac{\psi(x_{k+1}) - 2\,\psi(x_k) + \psi(x_{k-1})}{h^2})$$

$$\psi(x_0) = \psi(x_N) = 0$$

Sample potentials

$$V(x) = [V(-x)]^* \,, \qquad \psi(\pm 1) = 0.$$

$$V(x) = \begin{cases} +\mathrm{i}\, Z_n & x \in (-\ell_n, -\ell_{n-1}), \\ \\ -\mathrm{i}\, Z_n & x \in (\ell_{n-1}, \ell_n), \end{cases} ,$$

$n = 1, 2, \ldots, q + 1\,,$

$\ell_0 \, = 0 < \ell_1 < \ldots < \ell_{q+1} = 1.$

Equations: original

$$\left[-\frac{d^2}{dx^2} + V(x)\right] \psi(x) = E\,\psi(x).$$

and discretized

$$-\frac{\psi(x_{k+1}) - 2\,\psi(x_k) + \psi(x_{k-1})}{h^2}$$

$$= \mathrm{i}\,\mathrm{sign}(x_k) Z\,\psi(x_k) + E\,\psi(x_k)\,.$$

The Weigert's $N = 4$ matrix model

.

$$\left(\begin{array}{ccc} 2+\frac{1}{4}\,\mathrm{i}\,Z & -1 & 0 \\ & & \\ -1 & 2 & -1 \\ & & \\ 0 & -1 & 2-\frac{1}{4}\,\mathrm{i}\,Z \end{array} \right) \left(\begin{array}{c} \alpha_0 \\ \\ \gamma \\ \\ \beta_0 \end{array} \right) = \frac{1}{4}\,E \left(\begin{array}{c} \alpha_0 \\ \\ \gamma \\ \\ \beta_0 \end{array} \right)$$

.

easily generalized, (PTO)

$$
\left(\begin{array}{cccc|c|cc}
\mathrm{i}\xi - F & -1 & & & & & \\
-1 & \mathrm{i}\xi - F & \ddots & & & & \\
 & \ddots & \ddots & -1 & & & \\
 & & -1 & \mathrm{i}\xi - F & -1 & & \\
\hline
 & & & -1 & -F & -1 & \\
\hline
 & & & & -1 & -\mathrm{i}\xi - F & \ddots \\
 & & & & & -1 & \ddots \\
 & & & & & & \ddots
\end{array}\right)
$$

$$\psi = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \\ \hline \gamma \\ \hline \beta_n \\ \vdots \\ \beta_0 \end{pmatrix}$$

Solutions

$F = E\,h^2 - 2$ and $\xi = Z\,h^2$

$$\alpha_k = a_k + \mathrm{i}\,b_k, \qquad \beta_k = a_k - \mathrm{i}\,b_k \equiv \alpha_k^*,$$

$$\alpha_k = (a + \mathrm{i}b)\,U_k\left(\frac{-F + \mathrm{i}\xi}{2}\right), \qquad k = 0, 1, \ldots, n$$

$$U_k(\cos\theta) = \frac{\sin(k+1)\theta}{\sin\theta}.$$

Conditions

$$\gamma = (a + \mathrm{i}b)\ U_{n+1}\left(\frac{-F + \mathrm{i}\xi}{2}\right)$$

$$= (a - \mathrm{i}b)\ U_{n+1}\left(\frac{-F - \mathrm{i}\xi}{2}\right)$$

$$F\,\gamma = -\,(a + \mathrm{i}b)\ U_{n}\left(\frac{-F + \mathrm{i}\xi}{2}\right) -$$

$$-\,(a - \mathrm{i}b)\ U_{n}\left(\frac{-F - \mathrm{i}\xi}{2}\right)\ .$$

Robust solution at $F = 0$.

Parameter a **must vanish for even** $n = 0, 2, 4, \ldots$ **(and we may normalize** $b = 1$) **while** $b = 0$ **and** $a = 1$ **for the odd** $n = 1, 3, 5, \ldots$**.**

Secular equation in two alternative forms,

$$U_n\left(\tfrac{1}{2}\,\mathrm{i}\,\xi\right) - U_n\left(\tfrac{1}{2}\,\mathrm{i}\,\xi\right) = 0, \qquad n = 2m,$$

$$U_n\left(\tfrac{1}{2}\,\mathrm{i}\,\xi\right) + U_n\left(-\tfrac{1}{2}\,\mathrm{i}\,\xi\right) = 0, \qquad n = 2m + 1$$

satisfied identically at any $m = 0, 1, \ldots$**. QED.**

$$E = E_{n+2} = 2/h^2 = N^2/2 = 2(n+2)^2$$

Generic solutions at $F \neq 0$.

$$U_{n+1}\left(\frac{-F+\mathrm{i}\xi}{2}\right)(a+\mathrm{i}b) = U_{n+1}\left(\frac{-F-\mathrm{i}\xi}{2}\right)(a-\mathrm{i}b)$$

$$T_{n+1}\left(\frac{-F+\mathrm{i}\xi}{2}\right)(a+\mathrm{i}b) = -T_{n+1}\left(\frac{-F-\mathrm{i}\xi}{2}\right)(a-\mathrm{i}b)$$

define $(a,b) = (a_0, b_0)$ **and their ratio,**

$$T_{n+1}\left(\frac{-F+\mathrm{i}\xi}{2}\right)U_{n+1}\left(\frac{-F-\mathrm{i}\xi}{2}\right)+T_{n+1}\left(\frac{-F-\mathrm{i}\xi}{2}\right)U_{n+1}\left(\frac{-F+\mathrm{i}\xi}{2}\right) = 0.$$

$\Longrightarrow$ **the energies** F **as functions of the couplings** ξ**.**

Re-parametrization

$$\frac{-F+\mathrm{i}\xi}{2}=\cos\varphi, \qquad \operatorname{Re}\varphi=\alpha, \qquad \operatorname{Im}\varphi=\beta$$

i.e.,

$$\frac{1}{2}\,F=-\cos\alpha\cosh\beta, \qquad \frac{1}{2}\,\xi=-\sin\alpha\sinh\beta$$

and, in the opposite direction,

$$\cos\alpha=-\frac{1}{2\cosh\beta}\,F,$$

$$\sinh\beta=\frac{1}{2\sqrt{2}}\sqrt{F^2+\xi^2-4+\sqrt{(F^2+\xi^2-4)^2+16\,\xi^2}}.$$

gives trigonometric secular equation

$$\operatorname{Re} \frac{\sin[(n+1)\varphi] \cos[(n+1)\varphi^*]}{\sin \varphi} = 0.$$

Inspection.

In the domain with negative $\beta < 0$, roots

$\alpha \in (0, \pi/2)$ **at the negative $F < 0$, and**

$\alpha \in (\pi/2, \pi)$ **at the positive $F > 0$.**

The first roots in closed form,

$$F_0 = 0, \quad F_{\pm} = \pm\sqrt{2 - \xi^2}, \qquad n = 0,$$

$$F_0 = 0, \quad F_{\pm,\pm} = \pm\sqrt{2 - \xi^2 \pm \sqrt{1 - 4\xi^2}}, \qquad n = 1$$

etc. Critical values $Z = Z_{(crit)}(N)$ (PTO).

ALTERNATIVE OPTION:

Even dimensions $N-1=2n+2$:

$$
\left(\begin{array}{cccc|cccc}
\mathrm{i}\xi - F & -1 & & & & & \\
-1 & \mathrm{i}\xi - F & \ddots & & & & \\
 & \ddots & \ddots & -1 & & & \\
 & & -1 & \mathrm{i}\xi - F & -1 & & \\
\hline
 & & & -1 & -\mathrm{i}\xi - F & \ddots & \\
 & & & & \ddots & \ddots & -1 \\
 & & & & & -1 & -\mathrm{i}\xi - F
\end{array}\right)
\left(\begin{array}{c}
\alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \\ \hline \alpha_n^* \\ \vdots \\ \alpha_0^*
\end{array}\right) = 0.
$$

PARALLELISM.

Whenever the $\mathcal{PT}-$symmetry remains unbroken, closed solution follows from the matching conditions

$$(a+\mathrm{i}b)\;U_{n+1}\left(\frac{-F+\mathrm{i}\xi}{2}\right)=(a-\mathrm{i}b)\;U_n\left(\frac{-F-\mathrm{i}\xi}{2}\right)\,.$$

and

$$U_n\left(\frac{-F+\mathrm{i}\xi}{2}\right)\,U_n\left(\frac{-F-\mathrm{i}\xi}{2}\right)=U_{n+1}\left(\frac{-F+\mathrm{i}\xi}{2}\right)\,U_{n+1}\left(\frac{-F-\mathrm{i}\xi}{2}\right)\,.$$

Proof.

subproblem

$$\begin{pmatrix} -1 & \mathrm{i}\xi - F & -1 & 0 \\ \\ 0 & -1 & -\mathrm{i}\xi - F & -1 \end{pmatrix} \begin{bmatrix} U_{n-1}\left(\frac{-F+\mathrm{i}\xi}{2}\right)(a+\mathrm{i}b) \\ \\ U_{n}\left(\frac{-F+\mathrm{i}\xi}{2}\right)(a+\mathrm{i}b) \\ \\ U_{n}\left(\frac{-F-\mathrm{i}\xi}{2}\right)(a-\mathrm{i}b) \\ \\ U_{n-1}\left(\frac{-F-\mathrm{i}\xi}{2}\right)(a-\mathrm{i}b) \end{bmatrix} = 0$$

induces just one matching, $\alpha_{n+1} = \alpha_n^*$**. QED.**

MODELS WITH MORE MATCHING POINTS

$\ell = 1/2$ and $N = 6$ – analytic tractability

$$V(x) = \begin{cases} +\mathrm{i}\, Z & \\ 0 & \text{for} \quad x \in \begin{cases} \left(-1, -\frac{1}{2}\right), \\ \left(-\frac{1}{2}, \frac{1}{2}\right), \\ \left(\frac{1}{2}, 1\right) \end{cases} \\ -\mathrm{i}\, Z & \end{cases}$$

$$
\left(\begin{array}{c|ccc|c}
\mathrm{i}\xi - F & -1 & & & \\
\hline
-1 & -F & -1 & & \\
 & -1 & -F & -1 & \\
 & & -1 & -F & -1 \\
\hline
 & & & -1 & -\mathrm{i}\xi - F
\end{array}\right)
\left(\begin{array}{c}
\alpha_0 \\
\hline
\gamma_0 \\
\gamma \\
\gamma_0^* \\
\hline
\alpha_0^*
\end{array}\right) = 0.
$$

SUPPRESSING THE NON-HERMITICITY USING $\ell = 5/8$

$$V(x) = \begin{cases} +\mathrm{i}\, Z \\ \\ 0 \\ \\ -\mathrm{i}\, Z \end{cases} \qquad \text{for} \qquad x \in \begin{cases} \left(-1, -\frac{5}{8}\right), \\ \\ \left(-\frac{5}{8}, \frac{5}{8}\right), \\ \\ \left(\frac{5}{8}, 1\right) \end{cases}$$

$$
\left(\begin{array}{c|ccccc|c}
\mathrm{i}\xi - F & -1 & & & & & \\
\hline
-1 & -F & -1 & & & & \\
 & -1 & -F & -1 & & & \\
 & & -1 & -F & -1 & & \\
 & & & -1 & -F & -1 & \\
 & & & & -1 & -F & -1 \\
\hline
 & & & & & -1 & -\mathrm{i}\xi - F
\end{array}\right)
\left(\begin{array}{c}
\alpha_0 \\
\hline
\gamma_1 \\
\gamma_0 \\
\gamma \\
\gamma_0^* \\
\gamma_1^* \\
\hline
\alpha_0^*
\end{array}\right) = 0.
$$

secular determinant

$$
\mathcal{D} = \left[-F^6 - F^4\left(\xi^2 - 6\right) + F^2\left(4\,\xi^2 - 10\right) - 3\,\xi^2 + 4\right] F\,. \qquad (3)
$$

Roots remain real in the Hermitian $\xi \to 0$ limit (see Figure)

$$F_0 = 0, \qquad F_{\pm,0} = \pm\sqrt{2}, \qquad F_{\pm,\pm} = \pm\sqrt{2 \pm \sqrt{2}}, \qquad\qquad \xi \to 0.$$

$\xi_{crit} \approx 1.15470$**.**

five of the roots remain real for $Z \gg 1$,

$$F_0 = 0, \qquad F_{\pm,0} = \pm 1, \qquad F_{\pm,+} = \pm\sqrt{3}, \qquad\qquad \xi \to \infty.$$

$$
\left(\begin{array}{c|ccccc|c}
\mathrm{i}\xi - F & -1 & & & & & \\
-1 & \mathrm{i}\xi - F & -1 & & & & \\
\hline
 & -1 & -F & -1 & & & \\
 & & -1 & -F & -1 & & \\
 & & & -1 & -F & -1 & \\
\hline
 & & & & -1 & -\mathrm{i}\xi - F & -1 \\
 & & & & & -1 & -\mathrm{i}\xi - F
\end{array}\right)
\left(\begin{array}{c}
\alpha_0 \\
\alpha_1 \\
\hline
\gamma_0 \\
\gamma \\
\gamma_0^* \\
\hline
\alpha_1^* \\
\alpha_0^*
\end{array}\right) = 0
$$

$$\mathcal{D} = \left[-F^6 - F^4\left(2\,\xi^2 - 6\right) + F^2\left(-\xi^4 + 4\,\xi^2 - 10\right) + 2\,\xi^4 + \xi^2 + 4\right] F \tag{4}$$

PENTADIAGONAL REAL REFORMULATION

$$
\left(\begin{array}{cc|cc|cc|cc|c}
-F & -\xi & -1 & 0 & & & & & \\
\xi & -F & 0 & -1 & & & & & \\ \hline
-1 & 0 & -F & -\xi & \ddots & & & & \\
0 & -1 & \xi & -F & & \ddots & & & \\ \hline
 & & \ddots & & \ddots & \ddots & -1 & 0 & \\
 & & & \ddots & \ddots & \ddots & 0 & -1 & \\ \hline
 & & & & -1 & 0 & -F & -\xi & -1 \\
 & & & & 0 & -1 & \xi & -F & 0 \\ \hline
 & & & & & & -2 & 0 & -F
\end{array}\right)
\left(\begin{array}{c}
a_0 \\ b_0 \\ \hline a_1 \\ b_1 \\ \hline \vdots \\ \vdots \\ \hline a_n \\ b_n \\ \hline \gamma
\end{array}\right) = 0.
$$

$$\vec{\mathbf{c}}_{\mathbf{k}} = U_k\left(\frac{1}{2}\mathbf{X}\right)\,\vec{\mathbf{c}}_{\mathbf{0}}\,, \qquad \mathbf{X} = \begin{pmatrix} -F & -\xi \\ \xi & -F \end{pmatrix}, \qquad \mathbf{k} = 0, 1, \ldots, n+1\,.$$

SUMMARY: NEW HORIZONS:

- **A. possible new physics**

- **B. new feasible calculations**

Moreover:

ALSO FEASIBILITY:

- (a) **of proofs** (reality of energies)
  - (i) square-well $V(x)$ used (friendly math)
  - (ii) Runge-Kutta x used (friendly phys)

AND FEASIBILITY:

- (b) **of model building**
  - (i) “realistic” shapes of $V(x)$ (phys made useful)
  - (ii) “realistic” shapes of paths x (math made flexible)

CONCERNING ANALYTIC CONTINUATION:

- (a) **new mathematics of rigorous proofs**
  - (i) changing variables in SE (math kept friendly)
  - (ii) rectified x (SFQM phys made friendly)

CONCERNING ANALYTIC CONTINUATION:

- (b) **in model building** using tobogganic paths $\mathcal{C}^{(N)}$
  - (i) bound states (topology-dependent phys)
  - (ii) tobogganic scattering (math made challenging again)

last page: on non-Hermiticity, REMEMBER:

- **A. in physics**:

  I. unobservable coordinates (cf. KG),

  II. nonstandard scattering

- **B. in functional analysis**:

  I. analytic continuation,

  II. matching methods