PHYSICS NEAR EXCEPTIONAL POINTS

(or: towards systematics of quantum catastrophes)

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Chapter I. INTRODUCTION

EXCEPTIONAL POINTS?

definition 1 (the most elementary one):

The points of the loss of the reality

of **the least stable** pair

of the bound-state energies of H

♡ CHARACTERISTIC ODE MODEL:

see $E = k^2$ in M. Znojil, *PT-symmetric square well,* Phys. Lett. A 285 (2001) 7-10:

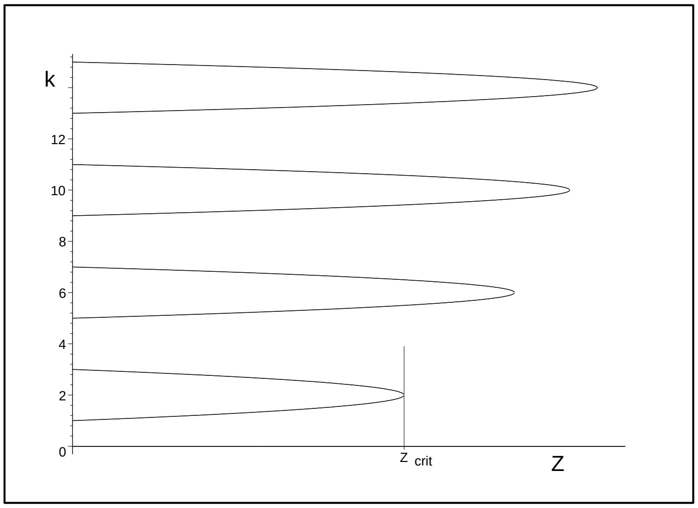

Figure 1: The least stable square-well energies are the lowest ones

$\diamondsuit$ AN EVEN MORE ELEMENTARY SCHEMATIC MATRIX MODEL:

$$\begin{pmatrix} -1 & b \\ \\ -b & 1 \end{pmatrix} \begin{pmatrix} \phi \\ \\ \chi \end{pmatrix} = E \begin{pmatrix} \phi \\ \\ \chi \end{pmatrix}$$

$$E = E_{\pm} = \sqrt{1 - b^2},$$
$$\Omega = \mathcal{D}^{(1)} = (-1, 1).$$

EXCEPTIONAL POINTS?

definition 2 ("in physics"):

certain points where "something is happening"

or, often, where "something goes wrong";

a typical "physical" EP: the $\alpha = 0$ trigger

of **the "fall on the center"** in $V(r) \sim \frac{\alpha^2 - 1/4}{r^2}$

♠ A SAMPLE OF EPs IN REAL LIFE

catastrophes in strong fields (Dirac equation: Greiner ’68),

complex EPs in perturbation theory (BW ’69: $\sqrt{\ldots}$ for AHOs),

in magnetohydrodynamics (Günther et al, this conference),

in nuclear physics: Scholtz et al, Heiss et al, Rotter et al,

in supersymmetric models: many authors,

in relativistic models: AM ’04 etc

etc.

♡ MODELS WITH EPs LOOK INNOVATIVE

physics can be **unusual** (non-local, superluminal, dissipative, ...)

one could **circumvent** no-go theorems (e.g., in supersymmetry)

relativistic (e.g., Proca) equations appear in a new perspective

MHD models = "physical" inside **as well as** outside Ω

last but, better, first, **field theory**

math. guide: EP in HO at $\alpha \to 0$ and all n:

$$E_n^+ = 4n + 2 + 2\alpha \quad \text{merges with} \quad E_n^- = 4n + 2 - 2\alpha,$$

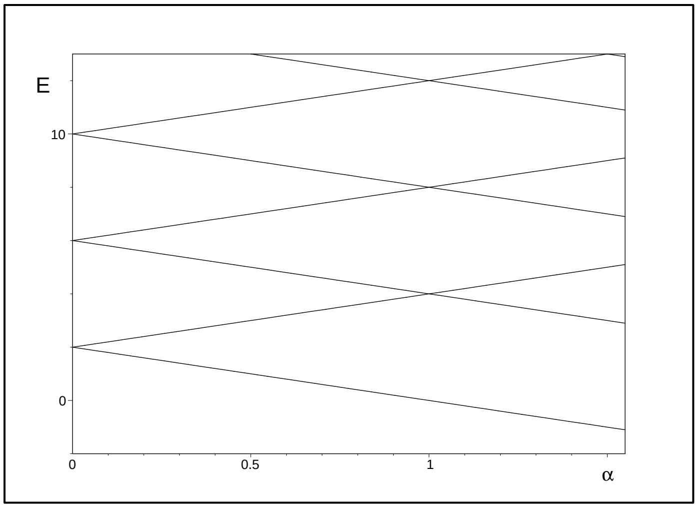

Figure 2: Spectrum of the PT-symmetric harmonic oscillator

♣ **Unfortunately**, HO is **NOT GENERIC:**

all the energies = **suddenly** complex **iff** $\alpha < 0$ (degeneracy)

"isolated" degenerate EPs $\exists$ **periodically** in α

all the energies = **linear** in α

LHO obtained at $\alpha = \frac{1}{2}$ (equidistance, SUSY etc)

M. Znojil, *PT-symmetric harmonic oscillators,*
Phys. Lett. A 259 (1999) 220 - 223).

EXCEPTIONAL POINTS?

definition 3 (*à la* T. Kato):

They form a boundary $\partial\Omega$

of the domain $\Omega = \Omega(H)$

of the quasi-Hermiticity of $H \neq H^\dagger$

an immediate task: *the determination of* $\partial\Omega$

$\Longrightarrow$ **CHALLENGE TO PHYSICS**

EPs are rather rare in Hermitian worlds with $H = H^\dagger$ (Heiss et al)

EP means a singularity in the metric Θ in quasi-Hermitian world

an immediate task: *the determination of* $\partial\Omega$

$\Longrightarrow$ **CHALLENGE TO MATHEMATICS**

♠ hard life beyond general four-by-four matrices $H^{[N]}$

M. Znojil, *Determination of the domain of the admissible matrix elements in the four-dimensional PT-symmetric anharmonic model,*

Phys. Lett. A, in print, online, quant-ph/0703168 (PTO).

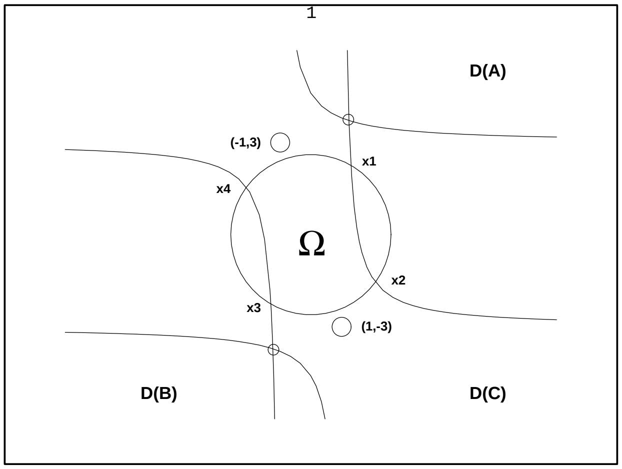

Figure 3: Graphical determination of the three-parametric domain $\Omega^{(4)} = \Omega^{(4)}(a, b, c)$ at a fixed $c = \sqrt{8/5}$, with the four "double" EPs x_1, x_2, x_3, x_4.

AN AMBITION OF THIS TALK

SIMPLIFY SOME **CURRENT HAMILTONIANS** $H \neq H^\dagger$
[say, $H = -\triangle + V(\vec{x})$, via a **discretization**, chapter II]

REVIEW BRIEFLY **THE GAINS**
(results on **matrices** $H = H^\ddagger = \Theta^{-1} H^\dagger\, \Theta$ inside Ω, ch. III,IV)

while trying to

formulate some **generic conjectures on** $\partial\Omega$
(method: symbolic manipulations plus extrapolations)

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Chapter II. DISCRETIZATIONS

A MENU

(1) RUNGE KUTTA QM
[**any** discrete **one-dimensional** $H = p^2 + V(x) \neq H^\dagger$]

(2) THE OTHER LATTICES OF COORDINATES
[2D here, semi-discrete, coupled channels, **square wells**]

(3) BIORTHOGONAL **BASES**
[mainly variational, **separably anharmonic oscillators**]

(1) RUNGE KUTTA QM IN 1D

coordinates: $x_k = x_{k-1} + h = -1 + kh,$

$$h = \frac{2}{N}, \qquad x_0 = -1, \qquad k = 1, 2, \ldots, N$$

kinetic energy:

$$-\psi''(x) \approx -\frac{\psi(x_{k+1}) - 2\,\psi(x_k) + \psi(x_{k-1})}{h^2}$$

boundary conditions: $\psi(x_0) = \psi(x_N) = 0$

leads to the Weigert's matrix square well model

.

$$\left(\begin{array}{ccc} 2+\frac{1}{4}\,{\rm i}\,Z & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2-\frac{1}{4}\,{\rm i}\,Z \end{array} \right) \left(\begin{array}{c} \alpha_0 \\ \gamma \\ \beta_0 \end{array} \right) = \frac{1}{4}\,E \left(\begin{array}{c} \alpha_0 \\ \gamma \\ \beta_0 \end{array} \right)$$

& its $N > 4$ generalizations **with tridiagonal** H,

(PTO)

$$
\left(\begin{array}{cccc|c|cc}
\mathrm{i}\xi - F & -1 & & & & & \\
-1 & \mathrm{i}\xi - F & \ddots & & & & \\
& \ddots & \ddots & -1 & & & \\
& & -1 & \mathrm{i}\xi - F & -1 & & \\
\hline
& & & -1 & -F & -1 & \\
\hline
& & & & -1 & -\mathrm{i}\xi - F & \ddots \\
& & & & & -1 & \ddots \\
& & & & & & \ddots
\end{array}\right)
$$

∃ AN EQUIVALENT **REAL FORMULATION OF SE:**

$$
\left(\begin{array}{cc|cc|cc|cc|c}
-F & -\xi & -1 & 0 & & & & & \\
\xi & -F & 0 & -1 & & & & & \\
\hline
-1 & 0 & -F & -\xi & \ddots & & & & \\
0 & -1 & \xi & -F & & \ddots & & & \\
\hline
 & & \ddots & & \ddots & \ddots & -1 & 0 & \\
 & & & \ddots & \ddots & \ddots & 0 & -1 & \\
\hline
 & & & & -1 & 0 & -F & -\xi & -1 \\
 & & & & 0 & -1 & \xi & -F & 0 \\
\hline
 & & & & & & -2 & 0 & -F
\end{array}\right)
\left(\begin{array}{c}
a_0 \\ b_0 \\ \hline a_1 \\ b_1 \\ \hline \vdots \\ \vdots \\ \hline a_n \\ b_n \\ \hline \gamma
\end{array}\right) = 0.
$$

SOLVABLE:

$\Longrightarrow$ by the matching method

$\Longrightarrow$ in closed **complex** form (Tschebyshev polynomials)

$\Longrightarrow$ in closed **real** form (matrix Tschebyshev)

M. Znojil (quant-ph/0605209),
Matching method and exact solvability of discrete
PT-symmetric square wells
J. Phys. A: Math. Gen. 39 (2006) 10247 - 10261

MERIT: A PARALLELISM

(i) solvable **differential** Schrödinger equation:

B. Bagchi et al (quant-ph/0503035),

PT-symmetric supersymmetry in a solvable short-range model,

Int. J. Mod. Phys. A 21 (2006) 2173-2190

(ii) parallel solvable **difference** Schrödinger equations:
[$N = 7$ **sample: PTO**]

$$
\left(\begin{array}{c|ccc|c}
\mathrm{i}\xi - F & -1 & & & \\
\hline
-1 & -F & -1 & & \\
 & -1 & -F & -1 & \\
 & & -1 & -F & -1 \\
\hline
 & & & -1 & -\mathrm{i}\xi - F
\end{array}\right)
\left(\begin{array}{c}
\alpha_0 \\
\hline
\gamma_0 \\
\gamma \\
\gamma_0^* \\
\hline
\alpha_0^*
\end{array}\right) = 0.
$$

$$
F_0 = 0, \qquad F_{\pm,\pm} = \pm\frac{1}{2}\sqrt{8 - 2\,\xi^2 \pm 2\,\sqrt{4 + \xi^4}}.
$$

$F_{\pm,-}$ responsible for EP, $\xi_{crit} = \sqrt{3/2} \approx 1.2247$ (PTO)

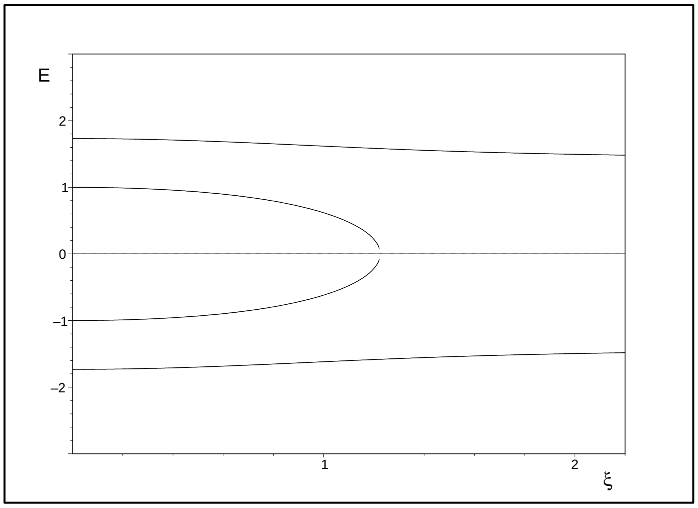

Figure 4: RK spectrum at $N = 7$, single fragile pair of $E_{1,3}$

$$F_{\pm,+} = \pm\sqrt{3}\left[1 - \frac{1}{12}\,\xi^2 + \frac{5}{288}\,\xi^4 + \frac{5}{3456}\,\xi^6 + O\left(\xi^8\right)\right]$$

$$F_{\pm,+} = \pm\sqrt{2}\left[1 + \frac{1}{4}\,y^2 - \frac{1}{32}\,y^4 - \frac{31}{128}\,y^6 + O\left(y^8\right)\right], \quad y = 1/\xi\,.$$

RUNGE KUTTA QM: SUMMARY

♡ feasible EP constructions

♢ a useful guidance towards $N \to \infty$

♠ oversimplified, tuned to 1D dynamics,

♣ not enough structural flexibility in O△E

$\Longrightarrow$ **idea**: try to move to 2D:

$(x, y) \longrightarrow (x, y_n)$ with $n = 1, \ldots, K$.

(2) COUPLED-CHANNEL QM

($K = 2$) two-by-two Hamiltonian, differential in x:

$$H_{(kinetic)} = \begin{pmatrix} -\frac{d^2}{dx^2} & 0 \\ 0 & -\frac{d^2}{dx^2} \end{pmatrix},$$

$$H_{(interaction)} = \begin{pmatrix} V_a(x) & W_b(x) \\ W_a(x) & V_b(x) \end{pmatrix}.$$

♡ 2D Hamiltonians which are only discretized in y

$\diamondsuit$ $\theta-$pseudo-Hermiticity:

$$\theta = \theta^\dagger = \left(\begin{array}{cc} 0 & \mathcal{P} \\ & \\ \mathcal{P} & 0 \end{array} \right) = \theta^{-1}$$

$\heartsuit$ square-well potentials $[x \in (-1, 0)]$:

$$\operatorname{Im} W_a(x) = X > 0,$$

$$\operatorname{Im} W_b(x) = Y > 0,$$

$$\operatorname{Im} V_a(x) = \operatorname{Im} V_b(x) = Z,$$

♠ spin-like ($\sigma = \pm 1$) symmetry:

$$\Omega = \left(\begin{array}{cc} 0 & \omega^{-1} \\ \\ \omega & 0 \end{array} \right), \qquad \omega = \sqrt{\frac{X}{Y}} > 0.$$

♣ **solvable and physical**

M. Znojil (quant-ph/0511085),

Coupled-channel version of PT-symmetric square well,

J. Phys. A: Math. Gen. 39 (2006) 441 - 455.

$(K = 3)$ next, three-by-three square well:

$$H_{(kinetic)} = \begin{pmatrix} -\frac{d^2}{dx^2} & 0 & 0 \\ 0 & -\frac{d^2}{dx^2} & 0 \\ 0 & 0 & -\frac{d^2}{dx^2} \end{pmatrix},$$

$$\mathbf{Z}_{(interaction,K=3)} = \begin{pmatrix} Z_{(1,1)} & Z_{(1,2)} & Z_{(1,3)} \\ Z_{(2,1)} & Z_{(2,2)} & Z_{(2,3)} \\ Z_{(3,1)} & Z_{(3,2)} & Z_{(3,3)} \end{pmatrix}.$$

with two “strong” $\theta-$pseudo-Hermiticities:

$$\theta = \theta_{(3,1)} = \left(\begin{array}{ccc} 0 & 0 & \mathcal{P} \\ \\ \mathcal{P} & 0 & 0 \\ \\ 0 & \mathcal{P} & 0 \end{array} \right) = \theta^{\dagger}_{(3,2)} = \theta^{-1}_{(3,2)} \neq \theta^{\dagger},$$

$$\theta_{(3,2)} = \left(\begin{array}{ccc} 0 & \mathcal{P} & 0 \\ \\ 0 & 0 & \mathcal{P} \\ \\ \mathcal{P} & 0 & 0 \end{array} \right) = \theta^{\dagger}_{(3,1)} = \theta^{-1}_{(3,1)} \neq \theta^{\dagger}_{(3,2)}.$$

♣ an emergence of a factorized symmetry

In the “strong” case we have $\theta^\dagger \neq \theta = \mathbf{R}$ in

$$H^\dagger = \mathbf{R}\, H\, \mathbf{R}^{-1}\,.$$

A few observations should be made

observation 1

if we wish that

$$H^\dagger = \mathbf{R}\, H\, \mathbf{R}^{-1}$$

then

$$H = \left(H^\dagger\right)^\dagger = \left(\mathbf{R}^{-1}\right)^\dagger \quad H^\dagger \quad \mathbf{R}^\dagger$$

observation 2

if we wish that

$$H^\dagger = \mathbf{R}\, H\, \mathbf{R}^{-1}$$

then

$$H = \left(H^\dagger\right)^\dagger = \left(\mathbf{R}^{-1}\right)^\dagger \quad H^\dagger \quad \mathbf{R}^\dagger$$

$$H = \left(H^\dagger\right)^\dagger = \left(\mathbf{R}^{-1}\right)^\dagger \mathbf{R}\, H\, \mathbf{R}^{-1} \mathbf{R}^\dagger$$

observation 3

if we wish that

$$H^{\dagger} = \mathbf{R}\, H\, \mathbf{R}^{-1}$$

then

$$H = \left(H^{\dagger}\right)^{\dagger} = \left(\mathbf{R}^{-1}\right)^{\dagger} \quad H^{\dagger} \quad \mathbf{R}^{\dagger}$$

$$H = \left(H^{\dagger}\right)^{\dagger} = \left(\mathbf{R}^{-1}\right)^{\dagger} \mathbf{R}\, H\, \mathbf{R}^{-1}\mathbf{R}^{\dagger}$$

$$H = \left(H^{\dagger}\right)^{\dagger} = \mathcal{S} \quad H \quad \mathcal{S}^{-1}$$

♠ **This means that**

our H must have a **factorized** symmetry,

$$H\,\mathcal{S} = \mathcal{S}\,H, \qquad \mathcal{S} = \left(\mathbf{R}^{-1}\right)^{\dagger}\mathbf{R}$$

♠ **This means that**

our H must have a **factorized** symmetry,

$$H\,\mathcal{S} = \mathcal{S}\,H, \qquad \mathcal{S} = \left(\mathbf{R}^{-1}\right)^{\dagger}\mathbf{R}$$

and we have two possibilities:

- **either** $\mathcal{S} = I$ (i.e., $\mathbf{R} = \mathbf{R}^{\dagger}$), *pseudo-Hermiticity*

♠ **This means that**

our H must have a **factorized** symmetry,

$$H\,\mathcal{S} = \mathcal{S}\,H, \qquad \mathcal{S} = \left(\mathbf{R}^{-1}\right)^{\dagger}\mathbf{R}$$

and we have two possibilities:

- **either** $\mathcal{S} = I$ (i.e., $\mathbf{R} = \mathbf{R}^{\dagger}$), *pseudo-Hermiticity*
- **or** $\mathcal{S} \neq I$ (proper *"factorized symmetry"*).

$\diamondsuit$ **factorized symmetry constraints** (for SQWs)

In three channels we get a **unique** 2-parametric set

$$\mathbf{Z}_{(interaction)} = \begin{pmatrix} Z & X & X \\ X & Z & X \\ X & X & Z \end{pmatrix}$$

and define $Z_{(eff)}(m)$ = eigenvalues of $\mathbf{Z}$,

$$Z_{eff}(1) = Z + 2\,X, \qquad Z_{eff}(2,3) = Z - X\,.$$

♡ **all = solvable again, via a 'new curve':**

$t = t^{(\sigma)}(s) = \frac{1}{2s} Z_{eff}(\sigma), \quad \sigma = 1, 2, 3$

[cf. M. Znojil (quant-ph/0601048),
Strengthened PT-symmetry with $P \neq P^\dagger$
Phys. Lett. A 353 (2006) 463 - 468].

new: degeneracy of levels $\sigma = 2, 3$,

EPs: a boundary $\partial\Omega^{(3)}$ of a triangle (PTO)

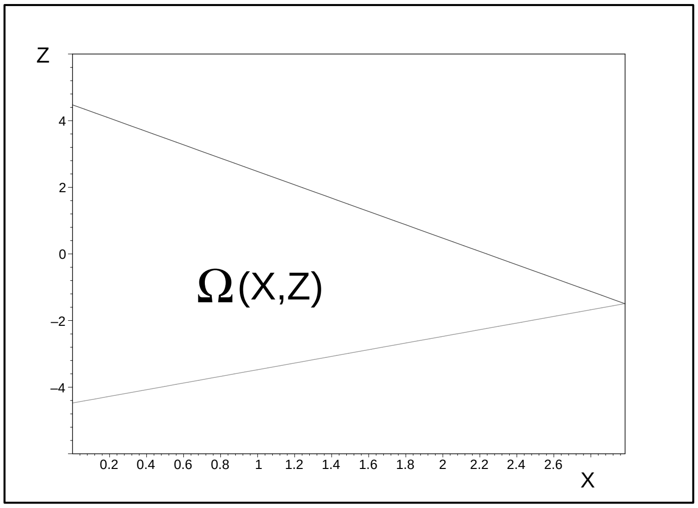

Figure 5: Triangular quasi-Hermiticity domain at $K = 3$

energies real for $X - Z_{crit} \le Z \le Z_{crit} - 2X$,

EP vertices $(0, \pm 4.475)$ and $(2.98, -1.49)$.

COUPLED-CHANNEL QM: SUMMARY

♡ an innovation of the concept of symmetries

♢ EPs from a routine square-well solvability

♠ an interpretation of $\mathcal{S} \neq I$ is missing,

♣ not worked out at the higher spins

$\Longrightarrow$ an **alternative** natural option:

a universal **biorthogonal basis**

(3) VARIATIONAL QM

initial Schrödinger equation

say, ODE with $x \in (-\infty, \infty)$, $\mathcal{PT}-$*symmetric:*

$$H = -\frac{d^2}{dx^2} + U(x) + \mathrm{i}\, W(x) \neq H^\dagger,$$

$$U(x) = U(-x)\,, \qquad W(x) = -W(-x)$$

represented in a **partitioned** variational basis,

$$| \psi_+ \rangle = \sum_{m=0}^{N_+} | 2m \rangle \, \phi_m, \quad | \psi_- \rangle = \sum_{m=0}^{N_-} | 2m+1 \rangle \, \chi_m$$

using the $\mathcal{PT}-$**symmetric** normalization,

$$| \psi \rangle = | \psi_+ \rangle - \mathrm{i} \, | \psi_- \rangle.$$

having to solve **two** Schrödinger equations at once:

$$H|n\rangle = E_n|n\rangle$$

and

$$\langle\!\langle n|H = E_n\langle\!\langle n|$$

we arrive at the **partitioned** infinite-dimensional

$$H = \begin{pmatrix} S & +B \\ \\ -B^T & L \end{pmatrix}$$

while $\mathcal{PT}-$**symmetry** implies that

$$|n\rangle = \begin{pmatrix} \vec{\phi}_n \\ \\ \vec{\chi}_n \end{pmatrix}, \qquad |n\rangle\!\rangle = \begin{pmatrix} \vec{\phi}_n \\ \\ -\vec{\chi}_n \end{pmatrix}.$$

$\diamondsuit$ *recipes: linear algebra:*

$$H^\dagger = \Theta\, H\, \Theta^{-1}, \qquad I \neq \Theta = \Theta^\dagger > 0.$$

with $H = \sum_n \, |n\rangle \, \frac{E_n}{\langle\langle n|n\rangle} \, \langle\langle n|$

and $\Theta = \sum_n \, |n\rangle\rangle \, t_n \, \langle\langle n|$

$\clubsuit$ *spectra: results, in general, numerical* illustration: $\exists$ **a nice extension**

of the above-mentioned two-by-two toy problem to dim=3,

$$H^{(3)} = \begin{pmatrix} -1 & a & d \\ \\ -a & 1 & b \\ \\ d & -b & 3+c \end{pmatrix}, \qquad \mathcal{P}^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ \\ 0 & -1 & 0 \\ \\ 0 & 0 & 1 \end{pmatrix}.$$

♡ in this model

Jacobi rotation annihilates $d = 0$,

$$H^{(3)} = \begin{pmatrix} -1 & a & 0 \\ \\ -a & 1 & b \\ \\ 0 & -b & 3 + c \end{pmatrix}$$

♡ ♡ **in this model**

all $c \neq -2$ are easy to incorporate in the formulae, and

a two-parametric "representative Hamiltonian" results,

$$H^{(3)} = \begin{pmatrix} -2 & a & 0 \\ -a & 0 & b \\ 0 & -b & 2 \end{pmatrix}$$

♡ ♡ ♡ **in this model**

the EP set $\partial\Omega^{(3)}$ has a **"generic"** shape,

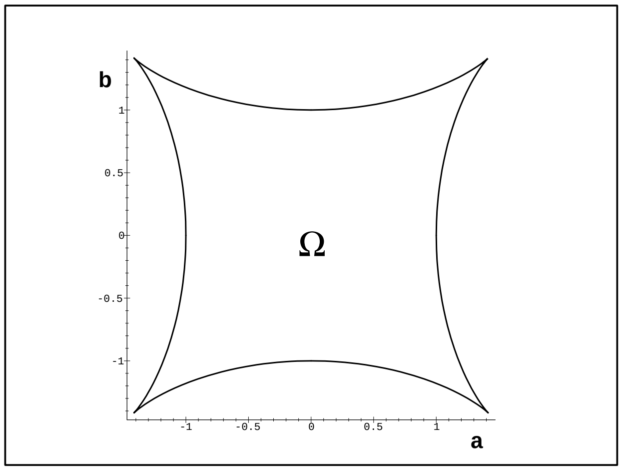

Figure 6: Domain $\Omega(a, b)$ for $H = H^{(3)}$

with two-dimensional **fixed$-b$ subdomains** $\Omega^{(2)}_{(b)}(a)$

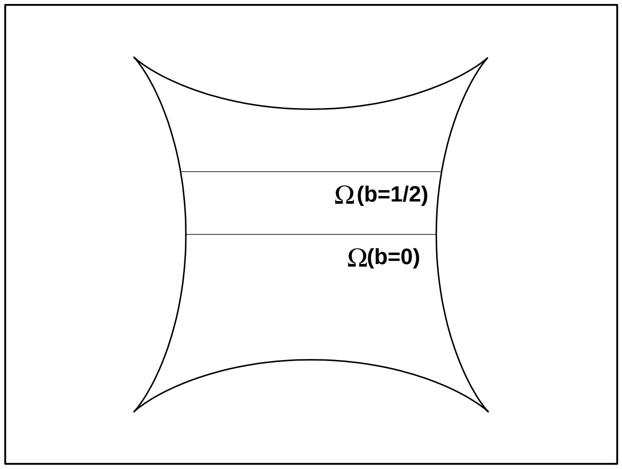

Figure 7: The $b-$dependence of the interior of $\Omega^{(3)}(a,b)$

$\heartsuit$ $\heartsuit$ $\heartsuit$ $\heartsuit$ **in this model**

conclude: **after a re-coupling**, $b = 0 \longrightarrow b > 0$,

EPs of a "decoupled system" $(= \partial\Omega^{(3)}(a, 0) = \pm 1)$,

become physical, **stabilized**, *"regularized"*

by having **moved inside** $\Omega^{(2)}_{(b)}(a) = (-\gamma(b), \gamma(b))$

where, e.g., $\gamma(1/2) \approx 1.04237$.

♡ ♡ ♡ ♡ ♡ **in this model**

$t \in (-1, 1)$ parametrizes the **whole** curve $\partial\Omega^{(3)}$,

$$a = a_{\pm} = \pm\sqrt{\frac{1}{2}\,(4 - 3t^2 - t^3)},$$

$$b = b_{\pm} = \pm\sqrt{\frac{1}{2}\,(4 - 3t^2 + t^3)}$$

(trick to be remembered!)

more details:

M. Znojil (quant-ph/0701232),
A return to observability near exceptional points in a schematic PT-symmetric model.
Phys. Lett. B 647 (2007) 225 - 230.

VARIATIONAL QM: SUMMARY

♡ straightforward calculations (linear algebra)

♢ reliable (variational) background at $N \gg 1$

♠ too numerical in general (too many free parameters)

♣ analytic insight ($N = 3$) looks exceptional

$\Longrightarrow$ **remedy**: choose some of the parameters in advance.

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Chapter III. **TRIDIAGONALIZATIONS:**

K coupled square wells in a chain

THE MENU

(1) RUNGE KUTTA
[tridiagonal by itself in 1D: omitted]

(2) COUPLED CHANNELS
[surprise: $\exists$ **circular-chain** semi-tridiagonalizations !]

(3) BIORTHOGONAL **BASES**
[**main** news on EPs, postponed to ch. IV]

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Section III. A. $K > 1$ **parallels to the single-channel case**

(i) $\exists$ formulae for wave functions:

square-well ODE with constant coefficients:

$$-\frac{d^2}{dx^2}\,\varphi^{(m)}(x) + \sum_{j=1}^{K} V_{Z_{(m,j)}}(x)\,\varphi^{(j)}(x) =$$

$$= E\varphi^{(m)}(x), \qquad m = 1, 2, \ldots, K$$

solvable by an ansatz

$$\varphi^{(m)}(x) = \begin{cases} C_L^{(m)} \sin \kappa_L (x+1), & x < 0, \\ \\ C_R^{(m)} \sin \kappa_R (-x+1), & x > 0 \end{cases}$$

using $Z^{(m)}_{(eff)}(K)$ as eigenvalues of

$$\begin{pmatrix} Z_{(1,1)} & Z_{(1,2)} & \dots & Z_{(1,K)} \\ Z_{(2,1)} & Z_{(2,2)} & \dots & Z_{(2,K)} \\ \vdots & \ddots & \ddots & \vdots \\ Z_{(K,1)} & Z_{(K,2)} & \dots & Z_{(K,K)} \end{pmatrix}.$$

(ii) quantized easily:

ansatz and its insertion:

$$\kappa_R = s + \mathrm{i}t = \kappa_L^*, \quad s > 0,$$

$$\rightarrow t = X_{first\ curve}\left[Z^{(m)}_{(eff)}(K), s\right] \equiv \frac{Z^{(m)}_{(eff)}(K)}{2s}$$

plus **matching** condition in the origin:

$$\rightarrow t = Y_{second\ curve}(s)$$

$Y_{second\ curve}(s)$: $\quad \kappa_L \operatorname{cotan} \kappa_L = -\kappa_R \operatorname{cotan} \kappa_R$

$\rightarrow$ implicit definition:

$$\boxed{2s\ \sin 2s + 2t\ \sinh 2t = 0}$$

$\rightarrow$ **energies**:

$$E_n = s_n^2 - t_n^2, \qquad n = 0, 1, \ldots .$$

$=$ **intersections** (s_n, t_n) of two graphs (PTO) *sample of the graphical recipe:*

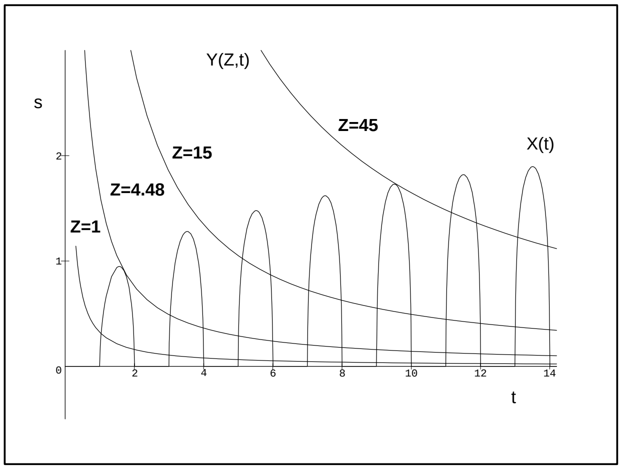

Figure 8: Square-well energies

(iii) the SAME algorithm determining EPs:

$= \text{hint} \to 2\,s = (n+1)\pi + (-1)^n\,Q$

$$\left.\frac{Q}{2}\right|_{crit} \equiv \varepsilon(t_{crit}) = \pi - \frac{Z_{crit}}{2t_{crit}},$$

$$\sin\left[2\,\varepsilon(t)\right] = \frac{t\ \sinh\ 2t}{\pi - \varepsilon(t)},$$

$\varepsilon_{(lower)}(t) = \pi/4$ and $\varepsilon_{(upper)}(t) = 0.$

Newton's method:

$$\partial_t \varepsilon(t_{crit}) = \frac{Z_{crit}}{2t_{crit}^2},$$

$$\partial_t \varepsilon(t) = \frac{\sinh\ 2t + 2t\ \cosh\ 2t}{2\ [\pi - \varepsilon(t)]\ \cos 2\varepsilon(t) - \sin 2\varepsilon(t)}$$

a sample result:

$\rightarrow t_{crit} \in (0.839393459, 0.839393461)$,

$\rightarrow s_{crit} \in (2.665799044, 2.665799069)$,

$\rightarrow E_{crit} \in (6.401903165, 6.401903294)$.

iteration	$Z_{crit}^{(lower)}$	$Z_{crit}^{(upper)}$
0	4.30	4.66
2	4.461	4.486
4	4.4743	4.4760
6	4.47524	4.47536
8	4.4753038	4.475312
10	4.47530826	4.47530882
12	4.47530856	4.47530861

(iv) formulae in **weakly** non-Hermitian regime:

$$s = s_n = \frac{(n+1)\pi}{2} + \tau\,\frac{Q_n}{2}, \quad \tau = (-1)^n$$

$\rightarrow$ **iterate**:

the first small quantity $\varrho \equiv \frac{1}{L} = \frac{1}{(n+1)\pi}$

the second one $\alpha = \frac{2\,Z_{eff}(\sigma)}{L}$ or $\beta = \alpha\varrho$

(v) at **intermediate** non-Hermiticities:

$\longrightarrow$ a “generalized continued fraction”

$$Q = \arcsin\left(2t\,\frac{\varrho}{1+\tau\,Q\,\varrho}\,\sinh 2t\right), \quad 2t = \frac{\alpha}{1+\tau\,Q\,\varrho}.$$

$\longrightarrow \arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \ldots$

$Q = Q(\alpha,\beta) = \alpha\beta\,\Omega(\alpha,\beta),$

$\longrightarrow \Omega(\alpha,\beta) = 1 + c_{10}\,\alpha^2 + c_{01}\,\beta^2 +$

$+ c_{20}\,\alpha^4 + c_{11}\,\alpha^2\beta^2 + c_{02}\,\beta^4 + \mathcal{O}(\alpha^6)$

$\rightarrow$ equation **re-arranged**:

$$[1 + \tau\,\beta^2\Omega(\alpha,\beta)]\,\text{arcsinh}(\Lambda) = \alpha$$

$$\Lambda = [1 + \tau\,\beta^2\Omega(\alpha,\beta)]^2\,\tfrac{1}{\beta}\,\sin[\alpha\beta\,\Omega(\alpha,\beta)]$$

(vi) formulae for energies:

→ leading order relation

$$0 = \left(-\tfrac{1}{6} + c_{10} + c_{01}\varrho^2 + 3\tau\varrho^2\right)\alpha^3 + \ldots .$$

determines the first two coefficients,

$$c_{10} = \tfrac{1}{6}, \qquad c_{01} = -3\tau\,,$$

the next-order $O\left(\alpha^5\right)$ gives

$c_{20} = \frac{1}{120}, \qquad c_{11} = \frac{1-8\tau}{6}, \qquad c_{02} = 15$

and leads to our **final** $1 + O\left(\alpha^4\right)$ formula

$$Q_n = \frac{4\,Z_{eff}^2}{(n+1)^3\pi^3} +$$

$$+ \frac{8\,Z_{eff}^4}{3\,(n+1)^5\pi^5}\left(1 + \frac{18\,(-1)^{n+1}}{(n+1)^2\pi^2}\right).$$

.

Section III. B. **There $\exists$ a semi-tridiagonalization of $2J+1$ coupled square wells**

At any K, let's use $\mathbf{R}^{-1} = \mathbf{R}^{\dagger}$, $\mathcal{S} = \mathbf{R}^2$

$$\mathbf{R} = \begin{pmatrix} 0 & \dots & 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & \dots & 0 & 0 \\ 0 & \mathcal{P} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \mathcal{P} & 0 \end{pmatrix}$$

in **FIVE channels,** *all* choices of $\mathbf{R}$ lead to *the same*

$$\mathbf{Z}_{(interaction)} = \begin{pmatrix} Z & X & D & D & X \\ X & Z & X & D & D \\ D & X & Z & X & D \\ D & D & X & Z & X \\ X & D & D & X & Z \end{pmatrix}$$

which is "next to tridiagonal" at $D = 0$,

$$\mathbf{Z}_{(interaction)} = \begin{pmatrix} Z & X & & & X \\ X & Z & X & & \\ & X & Z & X & \\ & & X & Z & X \\ X & & & X & Z \end{pmatrix}.$$

For 7 **channels**, similarly, the allowed matrix of couplings

$$\mathbf{Z}_{(interaction)} = \begin{pmatrix} Z & X & Y & D & D & Y & X \\ X & Z & X & Y & D & D & Y \\ Y & X & Z & X & Y & D & D \\ D & Y & X & Z & X & Y & D \\ D & D & Y & X & Z & X & Y \\ Y & D & D & Y & X & Z & X \\ X & Y & D & D & Y & X & Z \end{pmatrix}$$

becomes "almost tridiagonal" at $Y = D = 0$:

$$\mathbf{Z}_{(interaction)} = \begin{pmatrix} Z & X & & & & & X \\ X & Z & X & & & & \\ & X & Z & X & & & \\ & & X & Z & X & & \\ & & & X & Z & X & \\ & & & & X & Z & X \\ X & & & & & X & Z \end{pmatrix}.$$

COUPLED-CHANNEL CHAINS: SUMMARY

♡ an innovation of the concept of the chain model

♢ the solvability extended to the large “spins” $K = 2J + 1$

♠ not so nice at even $K = 2J$ (why? - an open question)

♣ no free parameters: $\partial\Omega^{(EP)}$ = piecewise linear

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Chapter IV.

SEPARABLY ANHARMONIC OSCILLATORS AND THEIR EXCEPTIONAL POINTS

THE MENU

(A) COUPLED HARMONIC-OSCILLATOR LEVELS
[$N < \infty$, an **auxiliary** upside-down symmetry]

(B) FULL, MAXIMAL CONFLUENCE OF EPs
[the results of **symbolic manipulations**]

(C) THE PATTERN OF DECONFLUENCE
[the shape of $\partial\Omega^{(N)}$, **strong-coupling regime**]

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Section IV. A. **The family of models**

The separable AHO bound-state problem with

$$H_{(int)} = \mathrm{i}\, g\ (\,|\,0\rangle\,\langle 1\,|+\ |\,1\rangle\,\langle 0\,|)$$

is equivalent to the above-mentioned matrix model

$$\left(\begin{array}{cc} -1 & g \\ & \\ -g & 1 \end{array} \right) \left(\begin{array}{c} \phi \\ \\ \chi \end{array} \right) = E \left(\begin{array}{c} \phi \\ \\ \chi \end{array} \right)$$

M. Znojil and H. B. Geyer (quant-ph/0607104),
Construction of a unique metric
in quasi-Hermitian quantum mechanics.
Phys. Lett. B 640 (2006) 52 - 56

For the same toy problem

$$H^{(2)} \begin{pmatrix} \phi \\ \\ \chi \end{pmatrix} = E \begin{pmatrix} \phi \\ \\ \chi \end{pmatrix}, \qquad H^{(2)} = \begin{pmatrix} -1 & a \\ \\ -a & 1 \end{pmatrix}$$

$\exists$ **a straightforward generalization (PTO):**

♡ **the chain-model family** ♡

$$H^{(N)} = \begin{bmatrix} 1-N & a & 0 & 0 & \dots & 0 \\ -a & 3-N & b & 0 & \dots & 0 \\ 0 & -b & 5-N & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & b & 0 \\ \vdots & \vdots & \ddots & -b & N-3 & a \\ 0 & 0 & \dots & 0 & -a & N-1 \end{bmatrix}.$$

♡ sample: the **four by four** secular equation

$$\det \begin{bmatrix} 3-E & b & 0 & 0 \\ -\,b & 1-E & a & 0 \\ 0 & -a & -1-E & b \\ 0 & 0 & -b & -3-E \end{bmatrix} = 0$$

with $a = \pm\sqrt{A}$, $b = \pm\sqrt{B}$ gives $E = \pm\sqrt{s}$.

This facilitates the study of EPs, e.g., via the *brute-force numerical determination of* $\partial\Omega^{(4)}$:

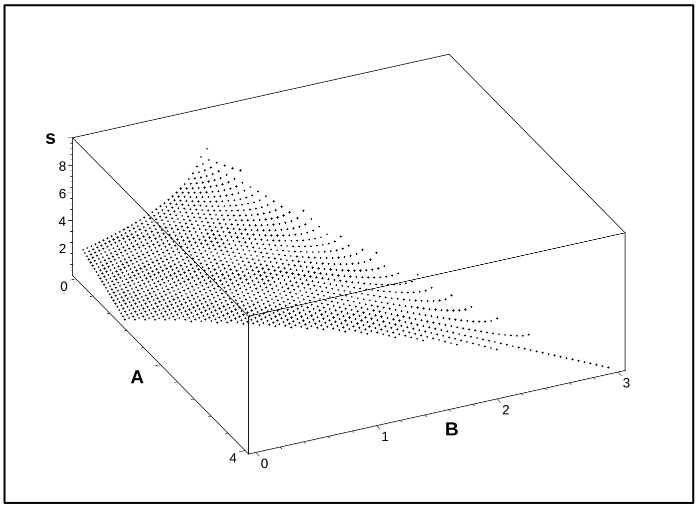

Figure 9: The least stable energy pair $E_{\pm} = \pm\sqrt{s(A,B)}$

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Section IV. B. **A** *maximal* **confluence of EPs**

Figure 9 indicates that the EP shape $\partial\Omega^{(4)}$ for

$$H^{(4)} = \begin{pmatrix} 3-E & b & 0 & 0 \\ -b & 1-E & a & 0 \\ 0 & -a & -1-E & b \\ 0 & 0 & -b & -3-E \end{pmatrix}$$

is **sharply spiked** near $a_{max}^{(EEP)} \approx 2$ and $b_{max}^{(EEP)} \approx \sqrt{3}$.

We intend to demonstrate that this feature is **generic.**

The $N = 4$ **EEP construction** using secular equation

$$s^2 + \left(-10 + 2\,b^2 + a^2\right) s + 9 + 6\,b^2 - 9\,a^2 + b^4 = 0$$

compared with $s^2 = 0$ at an extreme EP (EEP).

Gröbner-solvable two conditions:

$$A + 2\,B = 10, \qquad (3 + B)^2 = 9\,A$$

spurious solution: $A = 64$, $B = -27$,

the acceptable pair is unique: $A^{(EEP)} = 4$, $B^{(EEP)} = 3$.

EEP construction at any other N: **the method**:

(a) split the models $H^{(N)}$ in **two series:**

with $N = 2K$ *and with* $N = 2J + 1$

(b) construct explicit **Gröbner** EEP solutions

(at the first few N)

(c) **extrapolate** to all N

(and verify at the next few N).

$\diamondsuit \quad \spadesuit \quad \diamondsuit$ **the result:** $\diamondsuit \quad \spadesuit \quad \diamondsuit$

a re-parametrization of the couplings:

$$a^2 = G_1^{(N)}\,(1-\alpha)\ , \qquad b^2 = G_2^{(N)}\,(1-\beta)\ , \qquad \ldots,$$

$$G_k^{(N)} = k\,(N-k)\,.$$

with the innovated Greek-letter parameters $\in (0,1)$.

♣ **sampling the proof:** $N = 8$**:**

(1) **consider** secular equation

$$s^4 + P_3\, s^3 + P_2\, s^2 + P_1\, s + P_0 = 0$$

$\Longrightarrow$ $\Omega^{(8)}(A, B, C, D)$ circumscribed by the simplex

$$A + 2\,B + 2\,C + 2\,D = 84\,.$$

(2) **take** quadratic P_2, cubic P_1 and quartic P_0 containing 13, 19 and 20 individual terms, respectively, **reduce**, e.g.

$$P_2 = 1974 + (B + C + D)^2 + 2\,AD + 2\,BD + 2\,AC + \\ +50\,D - (83\,A + 142\,B + 70\,C)$$

& Gröbner-solve the EEP set of nonlinear equations

$$P_2(A,B,C,D) = 0,\;\; P_1(A,B,C,D) = 0,\;\; P_0(A,B,C,D) = 0$$

(3) **find and factorize** the resulting polynomial

$$314432\, D^{17} - 5932158016\, D^{16} + 4574211144896\, D^{15} +$$

$$+3133529909492864\, D^{14} + 917318495163561932\, D^{13} + \dots$$

$$\dots + 2353267541018244399368002288069050 73\, D^{2} -$$

$$-45376227941462117981555289702903979 7\, D +$$

$$+15371288194194653279861464836126516 7 = 0$$

(4) **demonstrate** that 16 roots are spurious:

(i) easy for all the complex roots;

(ii) easy for the three real but manifestly spurious negative roots

$D = -203.9147095, -156.6667001, -55.49992441.$

(iii) most complicated for the four real and positive roots

$D = 0.4192854385, 5.354156128, 1354.675195$ and $18028.16789.$

In the latter case, for example, one finds the spurious negative value for

$\Upsilon^2 \times A =$ (a polynomial in D of 16th degree).

No chance without computers: the number of digits in Υ^2 exceeds one hundred.

(5) **conclude** that the remaining closed solution

$$A^{(EEP)} = 16\,,\ B^{(EEP)} = 15\,,\ C^{(EEP)} = 12\,,\ D^{(EEP)} = 7$$

is unique.

QED.

Summary of the merits of the model:

- provides *all* types of the **confluence of EPs**

- *precisely* the necessary number of parameters.

More details:

M. Znojil (math-ph/0703070),
Maximal couplings in PT-symmetric chain-models with the real spectrum of energies.
J. Phys. A: Math. Theor. 40 (2007) 4863 - 4875

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Section IV. C.

The strong-coupling mechanism of the split of the EPs near EEPs

Let's recollect the **smoothness** of the surface $s(A, B) = E^2$,

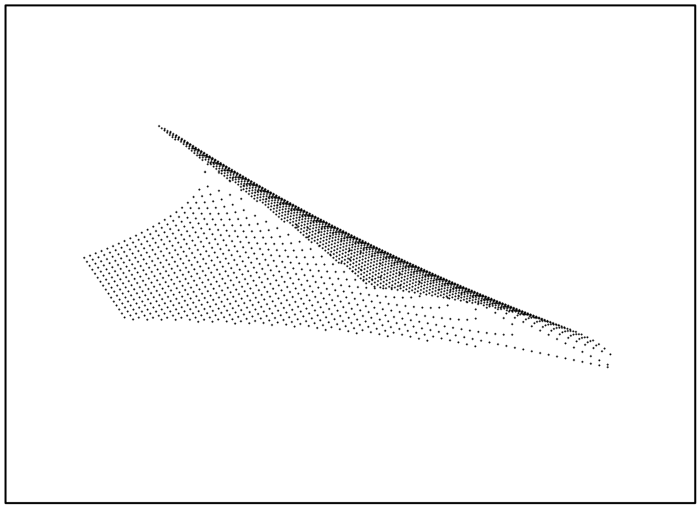

Figure 10: Second sheet of $s(A, B)\,(\leq 14)$ added to Fig. 9.

reparametrize the matrix elements of $H^{(4)}$ accordingly,

$$\begin{pmatrix} -3 & \sqrt{3}\,\sqrt{1-\beta} & 0 & 0 \\ -\sqrt{3}\,\sqrt{1-\beta} & -1 & 2\,\sqrt{1-\alpha} & 0 \\ 0 & -2\,\sqrt{1-\alpha} & 1 & \sqrt{3}\,\sqrt{1-\beta} \\ 0 & 0 & -\sqrt{3}\,\sqrt{1-\beta} & 3 \end{pmatrix}$$

and reparametrize **also its QH domain** $\Omega^{(4)}$**,**

$$\beta \geq \beta_{minimal} = \frac{9\,\alpha - \alpha^2}{9 + 3\,\alpha}\,, \qquad\qquad \alpha \in (0,1)$$

$$\alpha \geq \alpha_{minimal} = \beta - \frac{\beta^2}{4}\,, \qquad\qquad \beta \in (0,1)$$

$$\beta = t + t^2\,B(t)\,, \qquad \alpha = t + t^2\,A(t)$$

M. Znojil, *Conditional observability,*
Phys. Lett. B, to appear (arXiv:0704.3812v1 [hep-th] 28 Apr 2007)

◇ beyond the EEP we can get a "big bang" phenomenon,

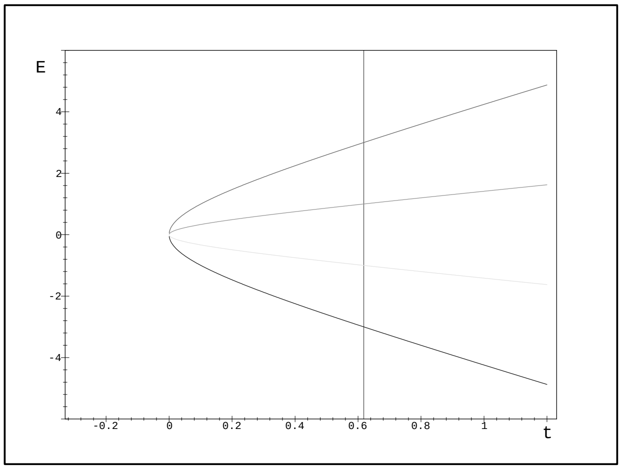

Figure 11: The $t-$dependence of the energy levels at $A = B = 1$.

Note that $\alpha = \beta = 1$ at the Hermiticity boundary

$t_{Herm.} = (\sqrt{5} - 1)/2 \approx 0.618033989.$

♢ ♢ at $A = 2$, $B = 1$ we get a non-EEP pattern,

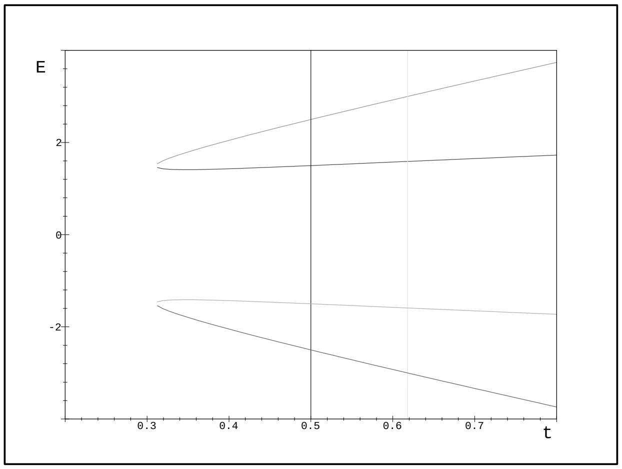

Figure 12: Two "simultaneous small bangs" at an EP value $t_{(EP)} > 0$. Note that the $\alpha(t) = 1$ line (i.e., the end of PT-symmetry) comes earlier than $\beta(t) = 1$ (i.e., the Hermiticity boundary).

$\diamondsuit \quad \diamondsuit \quad \diamondsuit$ at $A = 1,\ B = 1.5$ we have

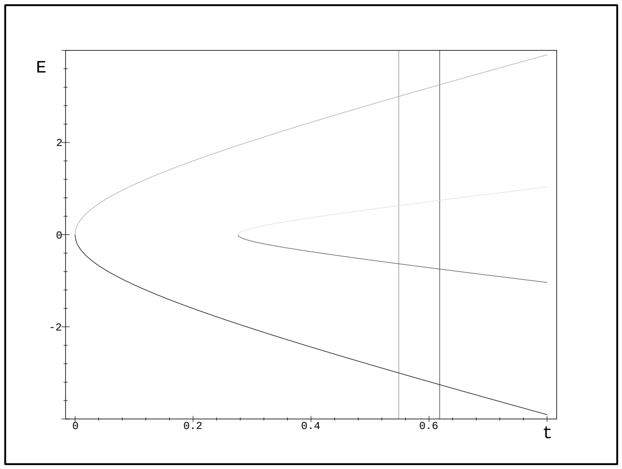

Figure 13: Quasi-Hermiticity established at $t_{(EP)} > 0.2$.

subconclusions: inside Ω,

$\exists$ **a fine-tuned balance** between $\alpha(t)$ and $\beta(t)$,

$\exists$ **all the possible** patterns of mergers between levels.

conjecture:

both properties are, *mutatis mutandis*, **valid at all** N

♡ test at the **six by six** $H^{(6)}$**,**

$$g_1 = c = \sqrt{5\,(1-\gamma)}\,, \quad g_2 = b = 2\,\sqrt{2\,(1-\beta)}\,,$$

$$g_3 = a = 3\,\sqrt{1-\alpha}$$

with

$$\alpha = t + t^2 + A\,t^3\,, \qquad \beta = t + t^2 + B\,t^3\,, \qquad \gamma = t + t^2 + C\,t^3\,.$$

◇ the "big bang" spectrum beyond the EEP again,

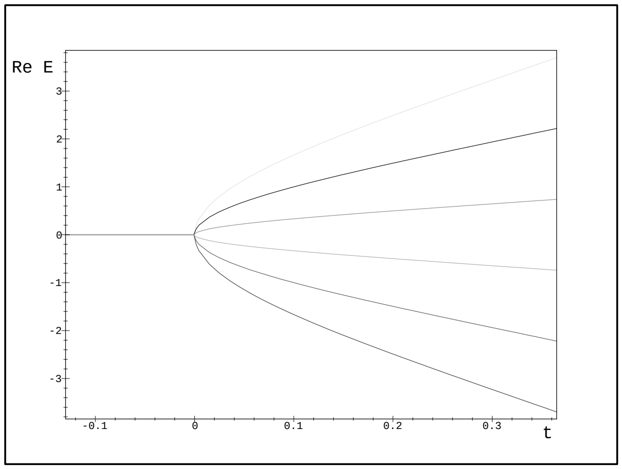

Figure 14: Energy levels at $A = B = C = 1$.

$\diamondsuit$ $\diamondsuit$ the choice of $A = 1,\ B = 2,\ C = 1$ gives

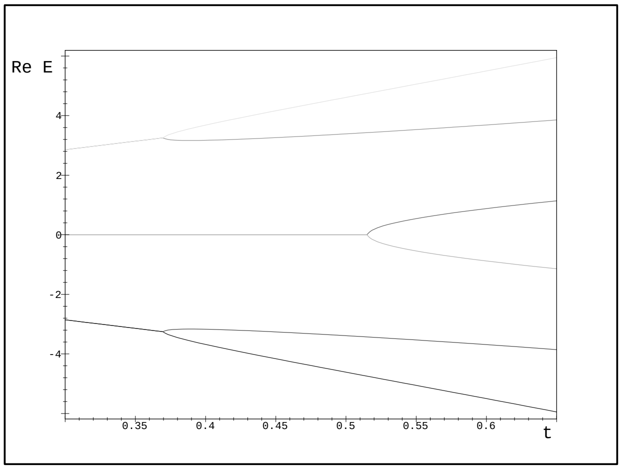

Figure 15: Real parts of the energies, "central" $t_{(EP)} > 0.5$.

$\diamondsuit \quad \diamondsuit \quad \diamondsuit$ at $A = 3,\ B = 5,\ C = 1$ one gets

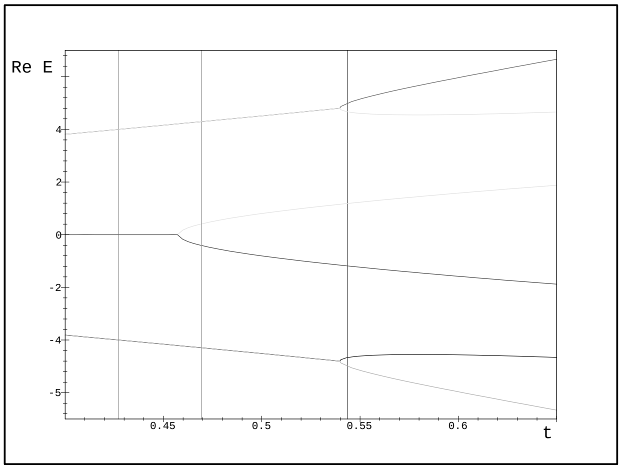

Figure 16: Quasi-Hermiticity established at the ground state.

conclusions: conjecture confirmed at $N = 6$:

$\exists$ **a fine-tuned balance** between α, β and γ

$\exists$ **all the possible** merging patterns again.

conjecture: at any $N = 2J\ (+1)$ **one should parametrize**

$$g_n = \sqrt{n\,(N-n)\,(1-\xi_n)}\,, \qquad \xi_n = t + t^2 + \ldots + t^{J-1} + G_n t^J\,,$$

$$n = 1, 2, \ldots, J\,.$$

SUMMARY OF THE MODEL:

IT IS

friendly: maximal (EEP) couplings = *integers*,

sufficient: *all* the EP-merging patterns are encountered

representative: its *symmetry* is "maximal"

necessary: *couldn't* manage with *less* free parameters

.

Chapter V.

IN PLACE OF AN OVERALL SUMMARY

FOUR KEY MESSAGES

to take home:

(α) *DISCRETIZATIONS* REVIEWED

(main merit: the variability of N)

(β) *CHAIN-MODELS* PROMOTED

(facilitated treatment, due to tridiagonality)

(γ) *BRUTE-FORCE CALCULATIONS* REPORTED

(an insight in EPs, based on symbolic manipulations)

(δ) *EXTRAPOLATIONS* PERFORMED/RECOMMENDED

(EP systematics)

END OF THE STORY

appendices

A BRIEF INTRODUCTION IN

QUASI-HERMITIAN QUANTUM MECHANICS

(1) Two Schroedinger equations in place of one,

$$H|n\rangle = E_n|n\rangle \; and \; \langle\langle n|H = E_n\langle\langle n|.$$

(2) Quasi-Hermiticity in place of Hermiticity,

$$H^\dagger = \Theta\, H\, \Theta^{-1}, \qquad I \neq \Theta = \Theta^\dagger > 0.$$

(3) A "‘generalized Dirac” notation,

$$H = \sum_n |n\rangle \frac{E_n}{\langle\langle n|n\rangle} \langle\langle n|$$

(4) The multiparametric **“choice of physics”**,

$$\Theta = \sum_n |n\rangle\rangle\, \theta_n \,\langle\langle n|\,, \qquad \theta_n > 0.$$

Illustration:

construction of the metric Θ for a 2×2 Hamiltonian,

$$H = \begin{pmatrix} -T & B \\ \\ -B & T \end{pmatrix}, \qquad \Theta = \begin{pmatrix} a & b \\ \\ b & d \end{pmatrix}$$

$$\Theta H = H^T \Theta \Longrightarrow 2bT = -B(a+d)$$

$$E \in I\!R \Longleftrightarrow |T| \geq |B|, \quad B = T \sin\alpha$$

ambiguity (the whole **interval** of a free parameter):

setting $T = 1$ and $a + d = 2Z > 0$ we have

$$H = \begin{pmatrix} -1 & \sin\alpha \\ \\ -\sin\alpha & 1 \end{pmatrix}, \qquad \Theta/Z = \begin{pmatrix} 1+\xi & \sin\alpha \\ \\ \sin\alpha & 1-\xi \end{pmatrix}$$

positivity $\theta_{1,2}/Z = 1 \pm \sqrt{\xi^2 + \sin^2\alpha} > 0$ means $1 > \sqrt{\xi^2 + \sin^2\alpha}$

solutions $\Theta = \Theta(\xi)$ are numbered by $0 < \xi < \cos\alpha$.

interpretation:

In 2D with biorthogonal “brabraket” basis,

$$\langle\langle n| \, H = \langle\langle n| \, E_n, \qquad H \, |n\rangle = E_n \, |n\rangle$$

ambiguity is compatible with the universal formula

$$\Theta = \boldsymbol{\Sigma} \; |n\rangle\rangle \, s_n \, \langle\langle n| \, , \qquad s_k > 0 \, .$$