Palma di Mallorca (seminar: Wednesday, May 30, 2007, 14.30)

Can we get observable, real spectra from non-Hermitian Hamiltonians?

or what should we all know about pseudo-Hermitian models in quantum mechanics

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A.

MOTIVATION

- *Let's ask: Do there exist non-Hermitian operators with real spectra?*
  - *The answer "Of course they do!" is easy.*

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a non-Hermitian spin-model illustrative prelude:

$$\left(\begin{array}{cc} s & b \\ -b & l \end{array} \right) \left(\begin{array}{c} \phi \\ \chi \end{array} \right) = E \left(\begin{array}{c} \phi \\ \chi \end{array} \right)$$

shift: $s = -1$, $l = 1$,

get: $E = E_{\pm} = \sqrt{1 - b^2}$,

set: $b = \cos\alpha = \sqrt{1 - E^2}$

- *But let's further ask: do they also find applications in quantum (and not only in quantum) physics?*
  - *the answer "Of course they do!" survives*

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proof: the famous Klein-Gordon equation

$$(i\,\partial_t)^2\,\Psi^{(KG)}(x,t) = \hat{H}^{(KG)}\,\Psi^{(KG)}(x,t)$$

with abbreviation

$$\varphi_1^{(PB)}(x,t) = i\,\partial_t\,\Psi^{(KG)}(x,t), \quad \varphi_2^{(PB)}(x,t) = \Psi^{(KG)}(x,t)$$

has the manifestly non-Hermitian Hamiltonian form,

$$i\,\partial_t \left(\begin{array}{c} \varphi_1^{(PB)}(x,t) \\ \\ \varphi_2^{(PB)}(x,t) \end{array} \right) = \hat{h}^{(PB)} \left(\begin{array}{c} \varphi_1^{(PB)}(x,t) \\ \\ \varphi_2^{(PB)}(x,t) \end{array} \right), \qquad \hat{h}^{(PB)} = \left(\begin{array}{cc} 0 & \hat{H}^{(KG)} \\ \\ 1 & 0 \end{array} \right).$$

Summarizing: non-Hermitian operators $H \neq H^\dagger$

- *can generate real spectra,*

- *do find applications in physics.*

OK - today, I'll detail these two answers a bit.

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A. I. Some more physics: Quantization under relativistic kinematics

- problems of physics with Klein-Gordon equation (indefinite norm)
- mathematics of Klein-Gordon equation (pseudo-Hermitian Hamiltonians)
- parallels with the Wheeler-De-Witt equation (cosmology)
- problems of physics with spin (unbounded Dirac's spectrum)
- strongly attractive Coulomb field (emergence of new degrees of freedom)
- problems in field theory (Gupta and Bleuler, Lee and Yang, Nishijima)

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A. II. Even more physics: Quantization under nonrelativistic kinematics

- mathematical puzzles of perturbation theory (complex couplings)
- Lipkin-Meshkov-Glick made non-Hermitian (Scholtz, Geyer and Hahne '92)
- quasi-Hermitian IBM models in nuclear physics (Dyson's bosonic mapping)
- particle(s) in a complex potential (indefiniteness of the norm, new SUSY)
- problems with coupled channels (unbounded spectrum, new degrees of freedom)
- beyond quantum theory (α^2 dynamos in MHD)

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B.
CONTEXT

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B. I.
Models with instabilities

- strong relativistic external fields (an abrupt complexification of E)
- too attractive central force (fall on the center, complexification of E)
- Jevicki-Rodrigues' limitations of SUSY (impenetrable singularities)
- Das-Pernice's' limitations of SUSY (imperfect regularization)
- 2D oscillators near cranking regime [$\omega_x = 3$, $\omega_x = 2$, $\Omega \in (2,3)$]
  (W. D. Heiss and R. Nazmitdinov, 2007)

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B. II.
Mathematics of the (in)stability

- (Dirac's Coulomb, etc): **antiparticles** or, in general, **physics** enters the scene

- (Bender-Turbiner '93): **analytic continuations** enter the scene (QES models)
- (Buslaev-Grecchi '93): $\mathcal{PT}-$**symmetry** enters the scene ($V(x) = -x^4$)
- (Bender-Boettcher '98): $\mathcal{PT}-$**symmetry** re-enters the scene ($V(x) = x^2(ix)^\varepsilon$)
- (Znojil '02, Mostafazadeh '02): **pseudo-Hermiticity** enters the scene
- (Znojil '06): the concept of $\mathcal{PT}-$**symmetric toboggans** enters the scene

C.
MATHEMATICS

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C. I..

Analytic continuation?

- (Bender and Wu '69): complexified **couplings**, $E_n(\lambda)$ = analytic function
- (Bender and Turbiner '93): $V(x) = a^2x^6 - 3ax^2$ = four eigenvalue problems
- (Cannata et al '98): exponential $V(x)$ = infinitely many eigenvalue problems
- (change of variables): spiked $V(x)$ = infinitely many eigenvalue problems

M. Znojil, Phys. Lett. A 342 (2005) 36 - 47

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C. II.

$\mathcal{PT}-$symmetry ?

- (Buslaev and Grecchi 1993): symmetry of $H = p^2 + \omega^2 x^2 - x^4$
- (Bender and Boettcher 1998): symmetry of $H = p^2 - (\mathrm{i}x)^{2+\varepsilon}$ with $\varepsilon \geq 0$
- (Mostafazadeh 2002): special case of $\mathcal{P}-$**pseudo-Hermiticity**
- (Znojil 2007): special case of **quantum toboggans**

J. Phys. A: Math. Gen. 39 (2006) 13325 - 13336

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D.
PSEUDOHERMITIAN QUANTUM MECHANICS

the spin-model prelude once more:

$$\begin{pmatrix} -1 & b \\ -b & 1 \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = E \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \qquad \mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

♡ **the pseudo-Hermiticity for pedestrians:** $H^\dagger = \mathcal{P}\, H\, \mathcal{P}^{-1}$

♣ **a restored Hermiticity for pedestrians:** $H^\dagger = \Theta\, H\, \Theta^{-1}$

ansatz for the "metric": $\Theta = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \Longrightarrow 2bT = -B(a+d)$

positivity of Θ: eigenvalues $\theta_{1,2} > 0 \Longleftrightarrow b \neq 0 \neq a + d = 2Z$

reparametrization: $a = Z(1+\xi)$, $d = Z(1-\xi)$,

family of acceptable metrics: $1 > \sqrt{\xi^2 + \sin^2\alpha}, \quad 0 \leq \xi < \cos\alpha$

the most popular choice: $\xi = 0$

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D. I.
Advanced definitions of pseudo-Hermiticity

- (textbooks): symmetry of H such that $H^\dagger = \mathcal{P}\, H\, \mathcal{P}^{-1}$, with $\mathcal{P} = \mathcal{P}^\dagger$
- (Mostafazadeh 2002): $\mathcal{P}$ need not be known
- (Solombrino 2002/ Znojil 2006): $\mathcal{P}$ need not be self-adjoint

  ($\Longrightarrow$ the "weak"/"strengthened" pseudo-Hermiticity)

  (M.Z., Phys. Lett. A 353 (2006) 463 - 468 and
  J. Phys. A: Math. Gen. 39 (2006) 4047 - 4061)

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D. II.

Typical example: ODE with $x \in (-\infty, \infty)$

$$H^{(BB)}(\nu) = -\frac{d^2}{dx^2} + g(x)\, x^2\,, \qquad g(x) = \omega^2 + (\mathrm{i}x)^\nu\,, \qquad \nu \geq 0$$

Hermiticity replaced by $\mathcal{PT}-$symmetry:

$$H = -\frac{d^2}{dx^2} + U(x) + \mathrm{i}\, W(x) \neq H^\dagger, \quad U(x) = U(-x)\,, \quad W(x) = -W(-x)\,,$$

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E.

ARE WE STILL *INSIDE* QUANTUM MECHANICS? YES, WE ARE!

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E. I.

Ad hoc redefinition of the inner product

- discovery (nuclear physics)

  (F.G.S., H.B.G. and F.J.W.H., Ann. Phys. 213 (1992) 74)

- rediscoveries: Znojil 2002, Mostafazadeh 2002, Bender et al 2002

- series of dedicated conferences:

  ($\Longrightarrow$ http://gemma.ujf.cas.cz/˜ znojil)

- various symbols for the "metric": T, $\mathcal{PQ}$, η_+, $\mathcal{CP}$, $\exp Q$, Θ

- the most recent review: C. Bender, hep-th/0703096.

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E. II. Quantum Mechanics on one page: Reality of spectra: condition *sine qua non*

- quintessence: **quasi-Hermiticity** $H^\dagger = \Theta\, H\, \Theta^{-1}$, $\Theta > 0$

- **explicit** formulae:

$H = \sum_n\, |n\rangle\, \frac{E_n}{\langle\langle n|n\rangle}\, \langle\langle n|$ and $\Theta = \sum_n\, |n\rangle\rangle\, t_n\, \langle\langle n|$

- **two** definitions needed:

$H|n\rangle = E_n|n\rangle$ **and** $\langle\langle n|H = E_n\langle\langle n|$

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Summary: All these results are freshly published:

Miloslav Znojil,

Maximal couplings in PT-symmetric chain-models with the real spectrum of energies

J. Phys. A: Math. Theor. 40 (2007) 4863 - 4875.

(math-ph/0703070).