.

$\mathcal{PT}$−symmetric quantum knots

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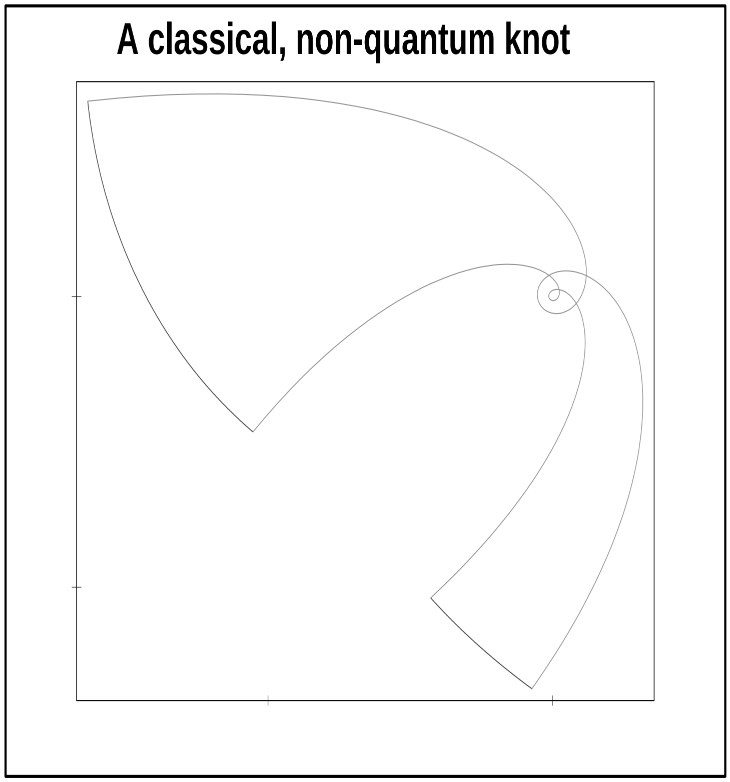
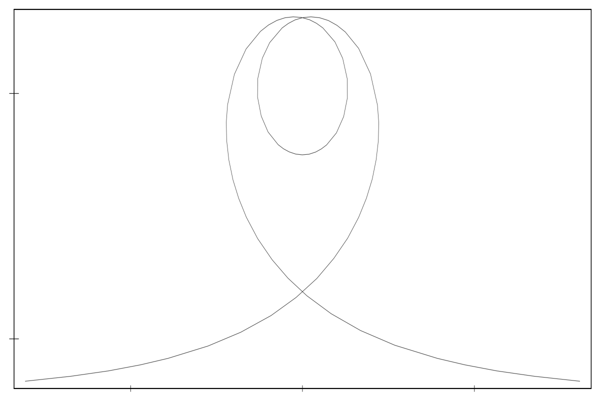

A classical, non-quantum knot

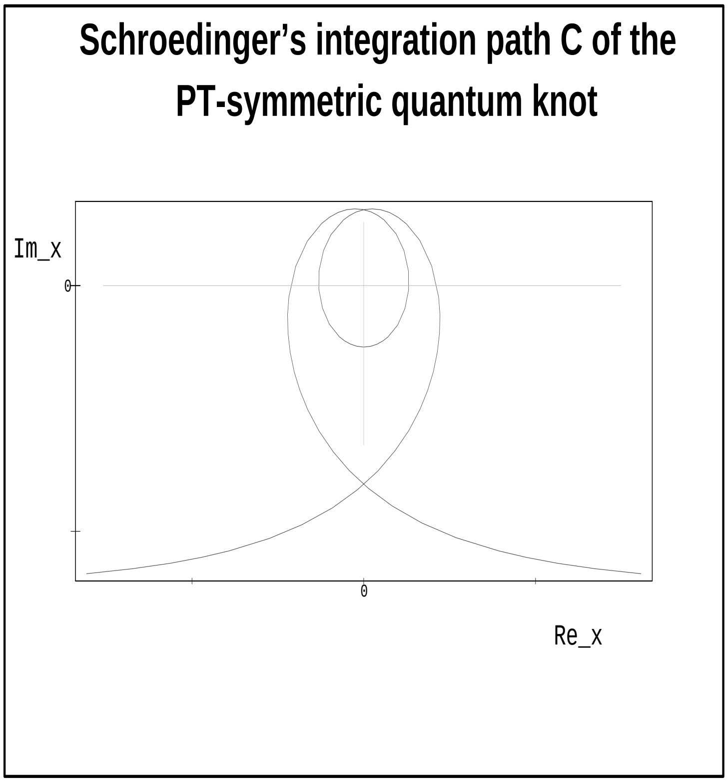
A classical knot which is
PT (= left-right) symmetric



OK, but what do I mean by a $\mathcal{PT}$−symmetric *quantum* knot?

A picture with a short answer:

P.T.O.


Schroedinger's integration path C of the
PT-symmetric quantum knot
Im_x
0
0
Re_x

The plan of my longer answer:

- *context:*

  schematic **Q(F)T** with $\psi \in \mathbb{R}$, $|\Psi(\psi)\rangle \in \mathbb{L}_2(\mathbb{R})$, i.e.,

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- **I.** *quantum knots* (elementary examples)

  (a) a potential-less $\mathcal{PT}-$symmetric quantum knot,

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- **II.** *quantum knots* (elementary examples)

  (a) a potential-less $\mathcal{PT}-$symmetric quantum knot,
  (b) quantum knots with $V \neq 0$

- **III.** *theory:* from $\mathcal{PT}-$S QM [Bender] to 3-HSF of QM

- **IV.** *new physics:* quasi-stationarity paradox resolved

II. QUANTUM KNOTS

(two or three elementary examples)

A. the first example:

• free radial Schrödinger equations with $n = 0, 1, \ldots$ in

$$-\frac{d^2}{dr^2}\,\psi(r) + \frac{\ell(\ell+1)}{r^2}\,\psi(r) = E\,\psi(r)\,, \qquad \ell = n + \frac{D-3}{2}$$

$E = \kappa^2$, $z = \kappa r$ and $\psi(r) = \sqrt{z}\,\varphi(z)$

• Bessel – solvable:

$$\psi(r) = c_1\,\sqrt{r}\,H_\nu^{(1)}(\kappa\,r) + c_2\,\sqrt{r}\,H_\nu^{(2)}(\kappa\,r)\,, \qquad \nu = \ell + 1/2\,.$$

asymptotic wedges = defined

on the multisheeted Riemann surface of multivalued analytic

wave functions $\psi(r)$

- $\mathcal{S}_0 = \{r = -\mathrm{i}\,\varrho\, e^{\mathrm{i}\,\varphi} \,|\, \varrho \gg 1\,,\ \varphi \in (-\pi/2, \pi/2)\}$,

- $\mathcal{S}_{\pm k} = \{r = -\mathrm{i}\, e^{\pm \mathrm{i}\, k\, \pi}\, \varrho\, e^{\mathrm{i}\,\varphi} \,|\, \varrho \gg 1\,,\ \varphi \in (-\pi/2, \pi/2)\}$,

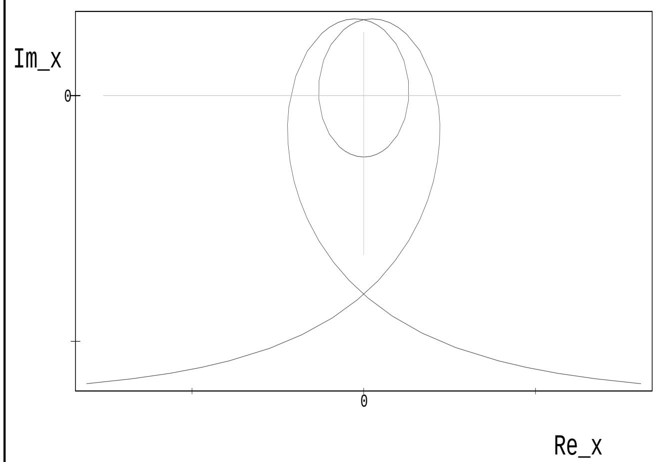
  k = 1, 2, ...

employed:

- knot-shaped integration contour $\mathcal{C}^{(N)}$

Remeber?

The path C with N = 2



employed:

- knot-shaped integration contour $\mathcal{C}^{(N)}$

- asymptotic formulae:

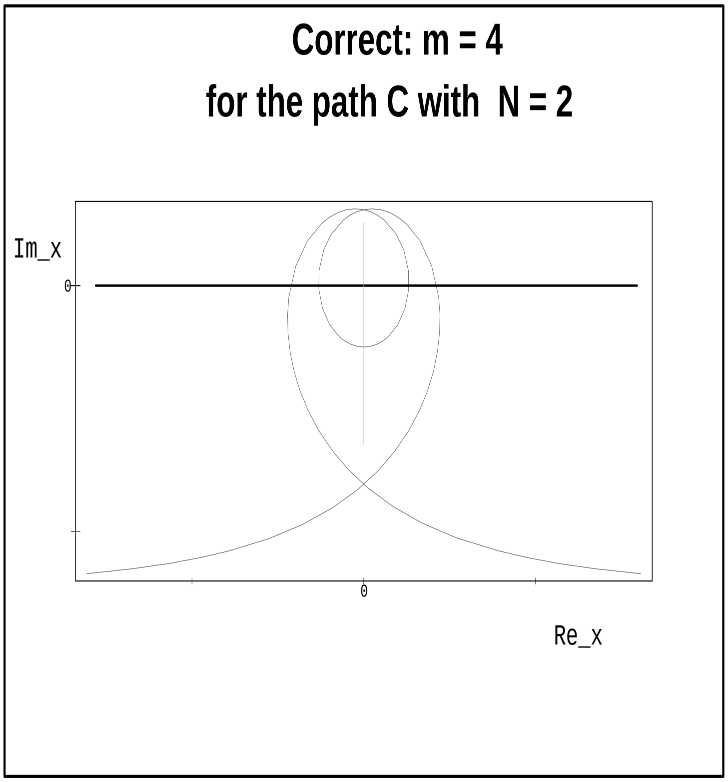
$$\sqrt{\frac{\pi z}{2}}\, H_\nu^{(1)}(z)\, \exp\left[-\mathrm{i}\left(z-\frac{\pi(2\nu+1)}{4}\right)\right] = 1-\frac{\nu^2-1/4}{2\mathrm{i}z}+\ldots\,,$$

$$\sqrt{\frac{\pi z}{2}}\, H_\nu^{(2)}(z)\, \exp\left[\mathrm{i}\left(z-\frac{\pi(2\nu+1)}{4}\right)\right] = 1+\frac{\nu^2-1/4}{2\mathrm{i}z}+\ldots\,.$$

- dichotomy: in $\mathcal{S}_{2k}$: unphysical $H_\nu^{(1)}(z)$, physical $H_\nu^{(2)}(z)$

  in $\mathcal{S}_{2k+1}$ physical $H_\nu^{(1)}(z)$, unphysical $H_\nu^{(2)}(z)$

solved, with $\mathcal{C}^{(N)}$ connecting $\mathcal{S}_0$ and $\mathcal{S}_m$, $m = 2N$:


Correct: m = 4
for the path C with N = 2
Im_x
0
0
Re_x

thus, with $\mathcal{C}^{(N)}$ connecting $\mathcal{S}_0$ and $\mathcal{S}_m$, $m = 2N$:

$$\psi(r) = c\,\sqrt{r}\,H_\nu^{(2)}(\kappa r) \quad (\text{with } r \in \mathcal{S}_0) \;\longrightarrow$$

$$H_\nu^{(2)}\left(z e^{\mathrm{i} m\pi}\right) = \frac{\sin(1+m)\pi\nu}{\sin\pi\nu}\,H_\nu^{(2)}(z) + e^{\mathrm{i}\pi\nu}\,\frac{\sin m\pi\nu}{\sin\pi\nu}\,H_\nu^{(1)}(z)$$

• **solved at any energy** $E = \kappa^2$, since

boundary conditions **quantize the angular momenta**:

$$2N\nu = integer\,, \qquad \nu \neq integer \;\Longrightarrow\; \ell = \frac{M-N}{2N}\,,$$

$$\boxed{M = 1, 2, 3, \dots\,, \ \textbf{with forbidden}\ M \neq 2N, 4N, 6N, \dots\,.}$$

$\odot$ SUMMARY:

• even dimensions $D = 2p \Longrightarrow \ell = n + p - 2 \Longrightarrow$

$M = (2n + 2p - 2)\, N$ **always forbidden**

• odd dimensions $D = 2p + 1 \Longrightarrow \ell = n + p - 3/2 \Longrightarrow$

$M = (2n + 2p - 1)\, N$ **never forbidden**

• monoenergetic finite-norm (i.e., wave-packet-like) solutions

• loss of the observability of the coordinates

B. the second example:

• same radial Schrödinger equation with new $V(r) = \gamma/r^2$ and with $n = 0, 1, \ldots$ in modified

$$\ell(\ell+1) = \gamma + \left(n + \frac{D-3}{2}\right)\left(n + \frac{D-1}{2}\right)$$

♡ **lemma: ∃ bound-state-supporting coupling constant**

$$\gamma = \left(\frac{M}{2N}\right)^2 - \left(n + \frac{D-2}{2}\right)^2$$

at any preselected dimension D, angular-momentum index n,

winding number N and an "allowed" integer M

C. the third example:

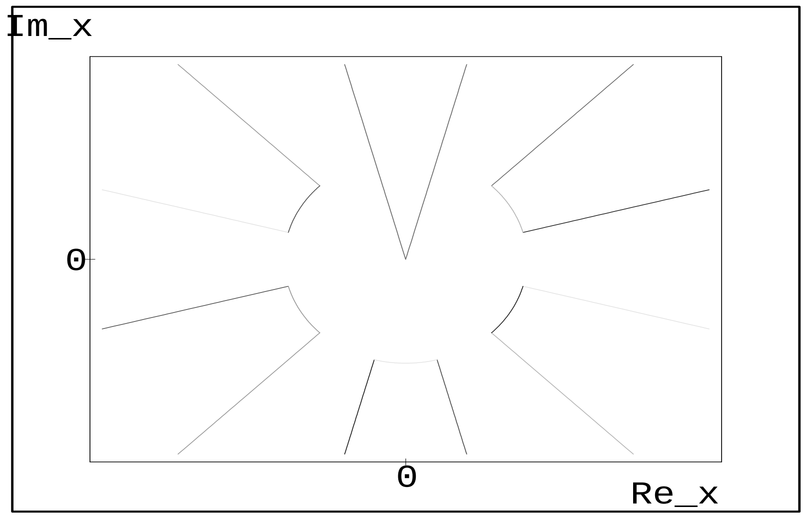
$$-\frac{d^2}{dr^2}\,\psi(r)+\frac{\ell(\ell+1)}{r^2}\,\psi(r)+V(r)\,\psi(r)=E\,\psi(r)$$

$$V(r)=r^2(\mathrm{i}\,r)^{\delta}+\alpha\,(\mathrm{i}\,r)^{\delta/2}$$

asymptotic wedges = defined differently

$$\psi(r)=c_1\,\psi^{(1)}(r)+c_2\,\psi^{(2)}(r)$$

$$\psi^{(1,2)}(r)=e^{\pm d\,r^f}+corrections\,,\qquad d=\frac{\mathrm{i}^{\delta/2}}{f}\,,\qquad f=2+\frac{\delta}{2}\,.$$



$$\mathrm{i}^{\delta/2} = \cos \tfrac{1}{4}\pi\delta + \mathrm{i}\, \sin \tfrac{1}{4}\pi\delta, \textbf{ decadic AHO } (\delta = 8)$$

$\bigoplus$ A CORRECT BUT NAIVE SUMMARY:

- a redefinition of the inner product in the Hilbert space is

needed – replace

$$\langle \psi \,|\, \phi \rangle = \int \psi^*(x)\phi(x)dx$$

by

$$\langle \psi \,|\, \phi \rangle = \int \psi^*(x)\,\Theta(x,y)\,\phi(y)dx\,dy\,.$$

.

III. THEORY:

three-Hilbert-space formulation of QM

A. back to textbooks

1. theoretical framework: standard QT

Table 1: Usual Dirac's notation in QM

Hilbert space	element	functional	inner product	Hamiltonian
$\mathcal{H}^{(1)}$	$\vert\psi\succ$	$\prec\psi\vert$	$\prec\psi\vert\psi'\succ$	$h = h^\dagger$

bra $\prec\psi|$**, ket** $|\psi\succ$

ON basis of eigenstates $\{|n\succ\}$

2. changes of representation $\mathcal{H}^{(1)} \longleftrightarrow \mathcal{H}^{(2)}$

employing **two** vector spaces : $\mathcal{H}^{(1)} = \Omega_{(unitary)}\, \mathcal{H}^{(2)}$

mappings of operators : $h = \Omega_{(unitary)}\, H\, \Omega^{-1}_{(unitary)}$

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= useful: Fourier transform $x \to p$: h_{kin} proportional to $-\triangle$

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= formal: ***the same*** **vector spaces:**

$\mathcal{H}^{(1)} = \mathbf{L}_2(\mathbb{R}^d)$ and $\mathcal{H}^{(2)} = \mathbf{L}_2(\mathbb{R}^d)$ (unitary equivalence).

3. main idea: use $\Omega_{(nonunitary)}$

and extend theoretical framework

Table 2: Adapted Dirac's notation

two Hilbert spaces	element	functional	inner product	Hamiltonian
$\mathcal{H}^{(1)}$	$\vert\psi\succ = \Omega\vert\psi\rangle$	$\prec\psi\vert = \langle\psi\vert\Omega^\dagger$	$\prec\psi\vert\psi'\succ$	$h = \Omega H \Omega^{-1}$
$\mathcal{H}^{(2)}$	$\vert\psi\rangle$	$\langle\psi\vert$	$\langle\psi\vert\psi'\rangle$	$H \neq H^\dagger$

B. special case: PTSQM

1. beyond unitarity:

$$\Omega \longrightarrow \Omega_{(nonunitary)} = \sum_{n,m=0}^{\infty} |\, n \succ\ \nu_{n,m} \,\langle\, m| \neq (\Omega^{-1}_{(nonunitary)})^{\dagger}$$

= invertible maps, $\mathcal{H}^{(1)} = \Omega_{(nonunitary)}\,\mathcal{H}^{(2)}$**,**

$$h = \Omega_{(nonunitary)}\, H\, \Omega^{-1}_{(nonunitary)}$$

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= invertible maps, $\mathcal{H}^{(1)} = \Omega_{(nonunitary)}\, \mathcal{H}^{(2)}$**,**

$$h = \Omega_{(nonunitary)}\, H \,\Omega^{-1}_{(nonunitary)}$$

= isospectrality with $h = \sum_{n=0}^{\infty} |\, n \succ \; E_n \; \prec n \,|$

$$H = \sum_{n=0}^{\infty} \Omega^{-1}_{(nonunitary)} \,|\, n \succ \; E_n \; \prec n \,|\, \Omega_{(nonunitary)} \,.$$

2. mathematics in $\mathcal{H}^{(2)}$:

= basis kets: $\quad |\, n \rangle := \Omega^{-1}_{(nonunitary)} |\, n \succ$

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= AND another set:

$$|\,n\rangle\!\rangle \;:= \Omega^{\dagger}_{(nonunitary)}\,|\,n\succ \equiv\; \Omega^{\dagger}_{(nonunitary)}\,\Omega_{(nonunitary)}\,|\,n\rangle \;\equiv\; \Theta\,|\,n\rangle$$

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= **updated spectral decomposition in $\mathcal{H}^{(2)}$:**

$$H = \sum_{n=0}^{\infty} |\,n\rangle\,E_n\,\langle\,n\,|\,\Theta \;\neq\; H^{\dagger}$$

= **"biorthogonal basis":** $\prec m|n\succ = \delta_{m,n} = \langle\!\langle m|n\rangle\,.$

3. physics in $\mathcal{H}^{(2)}$: SIMPLER Hs in

= nuclei: Dyson's $\Omega_{(nonunitary)}$ **[SGH '92]**

= molecules: generalized Fourier $\Omega_{(nonunitary)}$ **[BG '93]**

= fields: parity-pseudo-Hermiticity [BM '97, BB '98]

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$$\mathcal{P} := \sum_{n=0}^{\infty} \Omega^{\dagger}_{(nonunitary)} \,|\, n \succ\ \sigma_n\ \prec n \,|\, \Omega_{(nonunitary)}\,, \qquad \sigma_n = \pm 1\,,$$

= PTSQM: compatible with "the first principles"!

C. new: “3-HS QM”

preliminaries:

= a reduced ansatz $\Omega_{(nonunitary)} = \sum_{n=0}^{\infty} \,|\, n \succ \; \mu_n \,\langle\!\langle\, n|$

= the ambiguity of the metric is reproduced:

$$\Theta = \Omega^\dagger \, \Omega \; \equiv \; \sum_{n=0}^{\infty} \, |n\rangle\!\rangle \, \mu_n^* \, \mu_n \, \langle\!\langle\, n|$$

= the invertibility of the map and metric:

$$\Omega_{(nonunitary)}^{-1} = \sum_{n=0}^{\infty} \,|\, n\rangle \, \mu_n^{-1} \; \prec n| \,, \qquad \Theta^{-1} = \sum_{n=0}^{\infty} \, |n\rangle \, \frac{1}{\mu_n^* \, \mu_n} \, \langle\, n|$$

1. the third Hilbert space $\mathcal{H}^{(3)}$

Table 3: Definitions ($\Omega = \Omega_{(nonunitary)}$)

Hilbert space	element	**dual**	*inner product*	Hamiltonian
$\mathcal{H}^{(1)}$	$\|\psi\succ = \Omega\|\psi\rangle$	$\prec\psi\| = \langle\psi\|\Omega^\dagger$	$\prec\psi\|\psi'\succ$	$h = \Omega H \Omega^{-1}$ (Hermitian)
$\mathcal{H}^{(2)}$ (auxiliary)	$\|\psi\rangle$	$\langle\psi\|$	$\langle\psi\|\psi'\rangle$	H (non-Hermitian, simple)
$\mathcal{H}^{(3)}$	$\|\psi\rangle$	$\langle\!\langle\psi\| = \prec\psi\|\Omega$	$\langle\psi\|\Omega^\dagger\Omega\|\psi'\rangle$	H ([quasi-]Hermitian)

$$\mathcal{T}^{(2)} : |\psi\rangle \longrightarrow \langle\psi|\,, \qquad |\psi\rangle \in \mathcal{H}^{(2)}$$

$$\mathcal{T}^{(3)} : |\psi\rangle \longrightarrow \langle\!\langle\psi|\,, \qquad |\psi\rangle \in \mathcal{H}^{(3)}$$

2. remarks:

(a) $H = \Omega^{-1}_{(nonunitary)}\, h\, \Omega_{(nonunitary)}$ **can be Hermitian in** $\mathcal{H}^{(2)}$ **iff**

$h\, S = S\, h$ **where** $S = S^\dagger = \Omega_{(nonunitary)}\, \Omega^\dagger_{(nonunitary)} \neq I$

(b) $\langle\langle a|H\, b\rangle = \langle\langle a|H|b\rangle = \langle a|\Theta\, H|b\rangle \,\equiv\, \langle a|H^\dagger\, \Theta|b\rangle = \langle\langle H\, a|b\rangle$

(consistence of observability)

(c) the quasi-Hermiticity condition in $\mathcal{H}^{(2)}$**:**

$$H^\dagger = \Theta\, H\, \Theta^{-1}\,, \qquad \Theta := \Omega^\dagger_{(nonunitary)}\, \Omega_{(nonunitary)} \,\equiv\, \Theta^\dagger > 0$$

3. main theorem: unitary equivalence

. between $\mathcal{H}^{(3)}$ and $\mathcal{H}^{(1)}$

$$\langle\!\langle \psi_1 | \psi_2 \rangle = \prec \psi_1 | \, \Omega_{(nonunitary)} \, \Omega^{-1}_{(nonunitary)} | \psi_2 \succ \equiv \prec \psi_1 | \psi_2 \succ$$

A full parallel with Fourier transformation achieved.

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IV. NEW PHYSICS:

a sample of the broader applicability of the formalism:

quasi-stationarity paradox resolved

3-HS QM with time-dependent observables:

$$h(t) = \Omega(t)\, H(t)\, \Omega^{-1}(t)$$

(a) time-dependent Schrödinger equation in $\mathcal{H}^{(1)}$:

$$\mathrm{i}\, \partial_t |\varphi(t)\succ \;=\; h(t)\, |\varphi(t)\succ\,, \text{ solution } |\varphi(t)\succ \;=\; u(t)\, |\varphi(0)\succ$$

$$via \quad \mathrm{i}\partial_t u(t) = h(t)\, u(t)$$

(b) unitary in $\mathcal{H}^{(1,3)}$:

$$\prec \varphi(t) \,|\, \varphi(t) \succ \, = \, \prec \varphi(0) \,|\, \varphi(0) \succ , \;\; |\Phi(t)\rangle = \Omega^{-1}(t) \, |\varphi(t)\succ$$

$$\text{and} \;\; \langle\!\langle \Phi(t) \,| \, = \, \prec\varphi(t) \,|\, \Omega(t).$$

(c) feasible in $\mathcal{H}^{(3)}$ alone:

$$|\Phi(t)\rangle = U_R(t) \, |\Phi(0)\rangle \,, \qquad U_R(t) = \Omega^{-1}(t) \, u(t) \, \Omega(0)$$

$$|\Phi(t)\rangle\!\rangle = U_L^\dagger(t) \, |\Phi(0)\rangle\!\rangle \,, \qquad U_L^\dagger(t) = \Omega^\dagger(t) \, u(t) \, \left[\Omega^{-1}(0)\right]^\dagger \,.$$

MAIN THEOREM:

time-evolution generator in $\mathcal{H}^{(3)}$:

$$H_{(gen)}(t) = H(t) - \mathrm{i}\Omega^{-1}(t)\dot{\Omega}(t)$$

PROOF: via differential operator equations:

$$\mathrm{i}\partial_t U_R(t) = -\Omega^{-1}(t)\left[\mathrm{i}\partial_t \Omega(t)\right] U_R(t) + H(t)\, U_R(t)$$

$$\mathrm{i}\partial_t U_L^\dagger(t) = H^\dagger(t)\, U_L^\dagger(t) + \left[\mathrm{i}\partial_t \Omega^\dagger(t)\right] \left[\Omega^{-1}(t)\right]^\dagger U_L^\dagger(t)$$

In $\mathcal{H}^{(2)}$ they form the two non-Hermitian partners of the

standard evolution equation in $\mathcal{H}^{(1)}$.

verified also by the differentiation of the square of the norm:

$$
\begin{aligned}
\mathrm{i}\partial_t \langle\!\langle \Phi(t) \,|\, \Phi(t) \rangle = \mathrm{i}\partial_t \langle\!\langle \Phi(0) \,|\, U_L(t)\, U_R(t) \,|\, \Phi(0) \rangle = \\
= \langle\!\langle \Phi(0) \,|\, \left[\mathrm{i}\partial_t U_L(t)\right] U_R(t) \,|\, \Phi(0) \rangle + \langle\!\langle \Phi(0) \,|\, U_L(t) \left[\mathrm{i}\partial_t U_R(t)\right] \,|\, \Phi(0) \rangle = \\
= \langle\!\langle \Phi(0) \,|\, U_L(t) \left[-H(t) + \Omega^{-1}(t) \left[\mathrm{i}\partial_t \Omega(t)\right]\right] U_R(t) \,|\, \Phi(0) \rangle + \\
+ \langle\!\langle \Phi(0) \,|\, U_L(t) \left[H(t) - \Omega^{-1}(t) \left[\mathrm{i}\partial_t \Omega(t)\right]\right] U_R(t) \,|\, \Phi(0) \rangle = 0\,.
\end{aligned}
$$

QED.

IMPORTANT COROLLARY:

the time-dependent Schrödinger equations in $\mathcal{H}^{(3)}$:

$$\mathrm{i}\partial_t|\Phi(t)\rangle = H_{(gen)}(t)\,|\Phi(t)\rangle$$

$$\mathrm{i}\partial_t|\Phi(t)\rangle\!\rangle = H_{(gen)}(t)\,|\Phi(t)\rangle\!\rangle$$

where operator $H_{(gen)}(t)$ *is not* observable

$\odot$ SUMMARY:

- an additional dynamical information in the metric $\Theta \neq I$ (*ambiguity removal*)
- Θ allowed to depend on time: brachistochrone updates asked for
- $\Theta(t)$ tuned to ALL observables = *arbitrary* functions of time.

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THE END OF THE STORY

of PTSQM on Riemann sheets

with nontrivial monodromy group

and time-dependent Hilbert space $\mathcal{H}^{(3)}$

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SUPPLEMENTA AND APPENDICES

(bringing, first of all, references and historical remarks)

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(a) prehistory:

$\exists$ *complex* $V(x)$ *with real spectra:*

sample: Buslaev and Grecchi, 1993:

$V(x) \sim -x^4$ at $|x| \gg 1$:

exhibited $\mathcal{PT}-$symmetry

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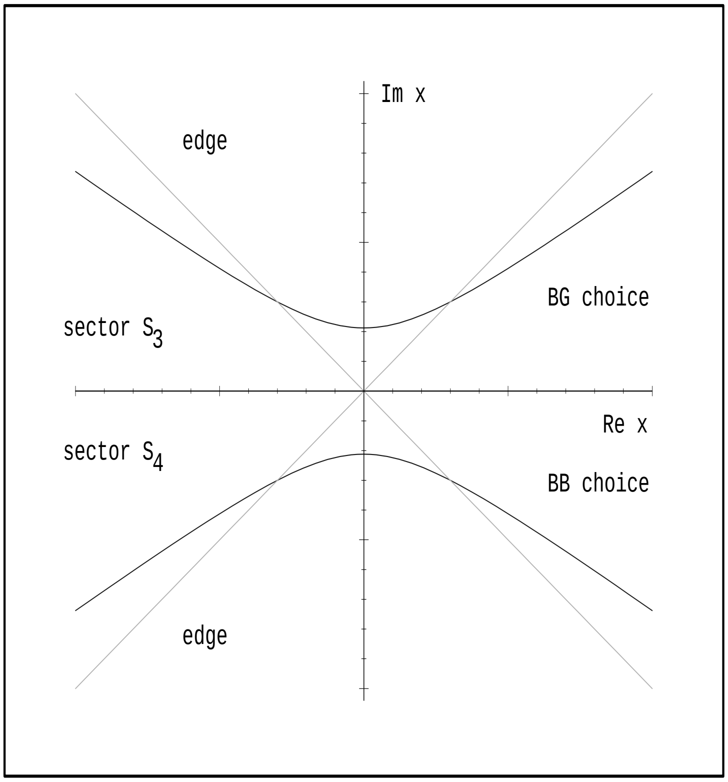

Figure 1: Complex curves of coordinates (BG oscillator)

explanation: $\exists$ a **Hermitian** equivalent of

$$H_{(\mathcal{PT})}\,\psi(x) = E\,\psi(x)$$

with Dirichlet abcs

$$\psi\left(\varrho \cdot e^{i\,\theta}\right) = 0, \qquad \varrho \gg 1$$

inside the wedges where, e.g.,

$$\theta_{left\,down} \in \left(-\frac{3\pi}{3}, -\frac{2\pi}{3}\right)$$

(Smilga: a “cryptoreality” of the spectrum)

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(b) 1998 = year zero:

class of BB's PT symmetric potentials:

$$V(x) = V_{symm}(x) + \mathrm{i}\, V_{antisymm}(x)$$

(c) 2001 = year one:

DDT's proofs of the reality of the spectra

.

(d) 2005 = the birth of QTs:

- MZ, quant-ph/0502041

  Phys. Lett. A 342 (2005) 36 - 47

- MZ, quant-ph/0606166

  J. Phys. A: Math. Gen. 39 (2006) 13325 - 13336

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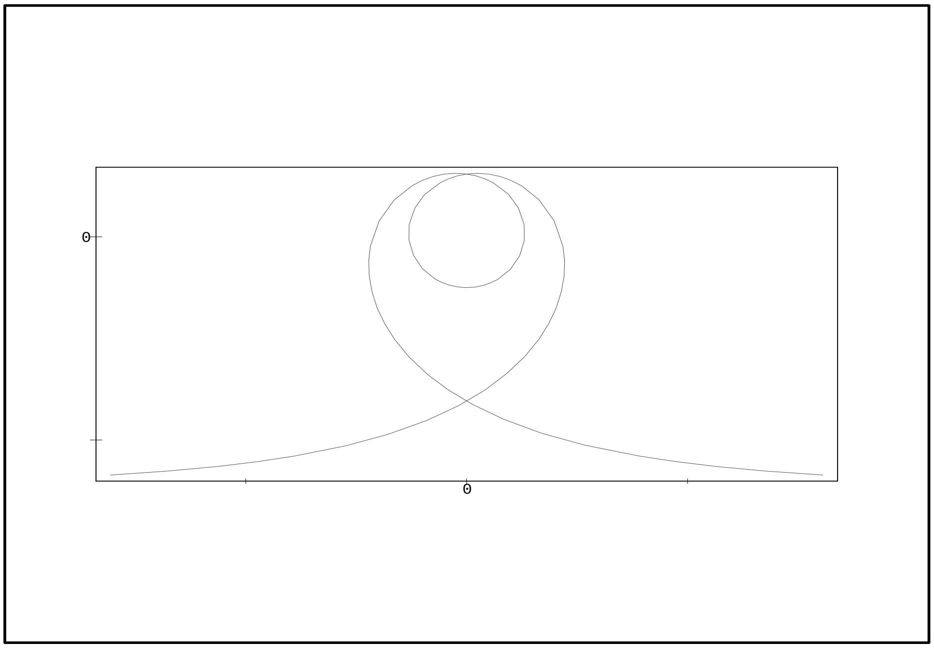

Figure 2: BG-oscillator toboggan at $\ell \neq -1, 0$, with $\mathcal{N} = 2$

II. MODELS ON COMPLEX CONTOURS $\mathcal{C}^{(N)}$:

$\mathcal{PT}$−symmetric quantum mechanics

(a) the first step: spiked HOs

MZ,

PT symmetric harmonic oscillators

Phys. Lett. A 259 (1999) 220 - 3.

$$\left(-\frac{d^2}{dx^2}+\frac{\ell(\ell+1)}{x^2}+x^2\right)\,\psi(x)=E\,\psi(x)$$

defined along straight contour

$$\mathcal{C}^{(0)} = \{x \mid x = t - i\,\varepsilon,\, t \in I\!R\}$$

$\exists$ "twice as many" bound-state levels

$$E = E_{n,\ell,\pm} = 4n + 2 \pm 2\alpha(\ell)$$

= *toboganically trivial*

(b) the second step: AHOs

(b.1) non-tobogganic abcs:

$$\left[-\frac{d^2}{dx^2} + V_{\mathcal{PT}}(x)\right] \psi(x) = E\,\psi(x)$$

$$\psi(\pm\mathrm{Re}\ L + i\,\mathrm{Im}\ L) = 0\,,$$

$$|L| \gg 1 \quad \text{or} \quad |L| \to \infty\,.$$

(b.2) tobogganic, along loops

on multisheeted Riemann surfaces

with, say, $\varphi \in (-(N+1)\pi,\ N\pi)$ in

$$\mathcal{C}^{(N)} = \left\{ x = \varepsilon\, \varrho(\varphi, N)\, e^{i\,\varphi}\,, \varepsilon > 0 \right\}\,.$$

$$\varrho(\varphi, N) = \sqrt{1 + \tan^2 \frac{\varphi + \pi/2}{2N+1}}$$

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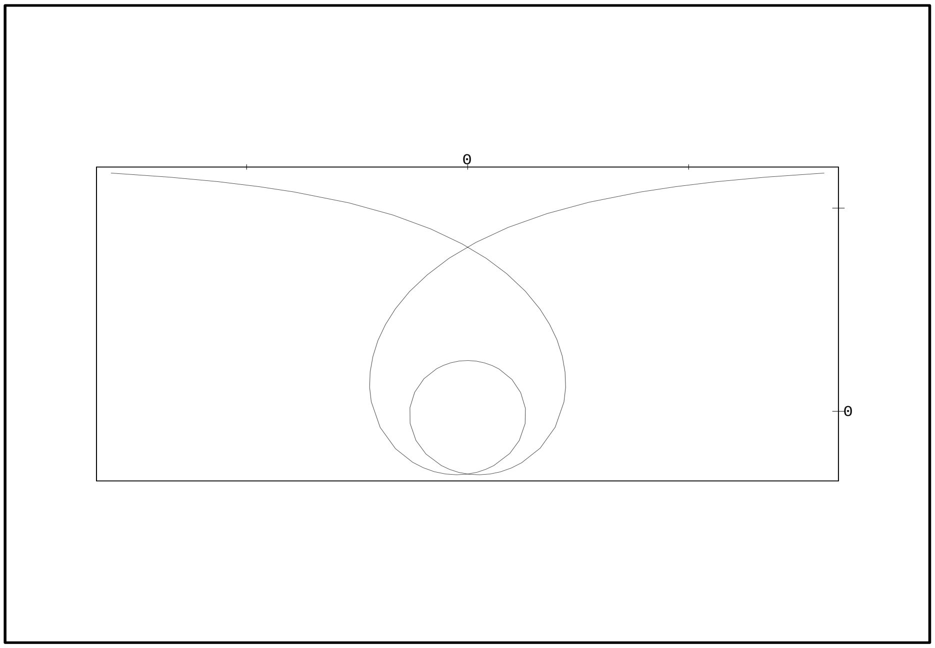

Figure 3: Upside down? Winding number still $\mathcal{N} = 2$

necesssary: branch point in $\psi(x)$, say, from

$$V(x) \sim \frac{irrational\ constant}{x^2}$$

what is then the $\mathcal{PT}-$symmetry of $\psi(x)$?

the left-right symmetry of $\mathcal{C}^{(N)}$

along the whole Riemann surface.

III. CONSTRUCTIONS OF QUANTUM TOBOGGANS

(a) QES models

Miloslav Znojil (quant-ph/0502041):

PT-symmetric quantum toboggans

Phys. Lett. A 342 (2005) 36-47.

model:

$$V(x) = x^{10} + \text{asymptotically smaller terms}$$

$$\psi(x) = e^{-x^6/6 + \text{asymptotically smaller terms}}$$

reparametrized

$$\psi(x) = \exp\left[-\frac{1}{6}\varrho^6 \cos 6\varphi + \dots\right],$$

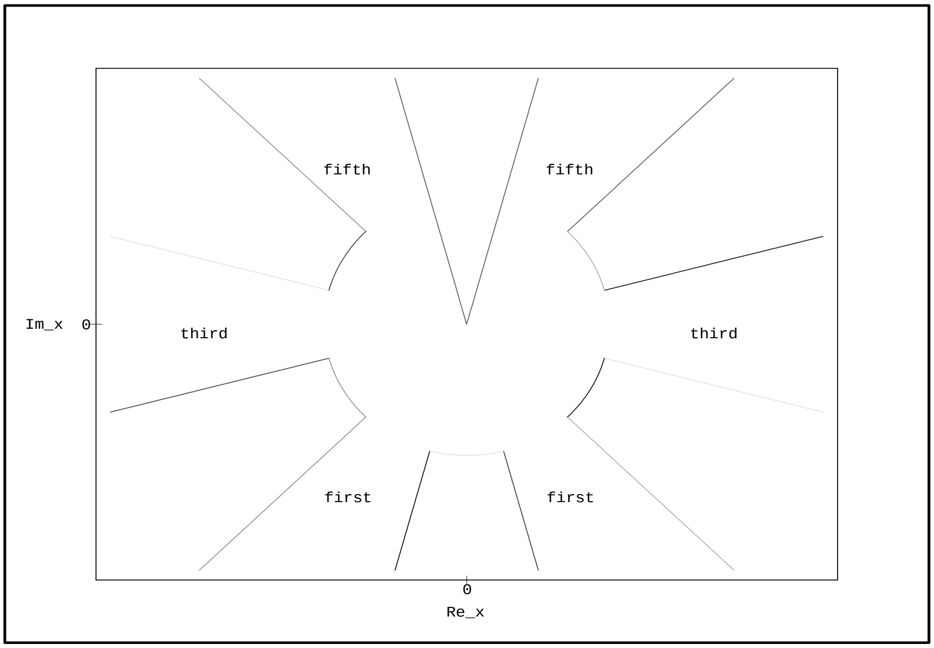

Figure 4: Domain of allowed asymptotics of decadic-oscillator contours

$\exists$ five non-tobogganic wedges:

$$\Omega_{(first\ right)} = \left(-\frac{\pi}{2} + \frac{\pi}{12}, -\frac{\pi}{2} + \frac{3\pi}{12}\right),$$

$$\Omega_{(first\ left)} = \left(-\frac{\pi}{2} - \frac{\pi}{12}, -\frac{\pi}{2} - \frac{3\pi}{12}\right),$$

$$\Omega_{(third\ right)} = \left(-\frac{\pi}{2} + \frac{5\pi}{12}, -\frac{\pi}{2} + \frac{7\pi}{12}\right), \quad \ldots$$

$$\ldots \quad \Omega_{(fifth\ left)} = \left(-\frac{\pi}{2} - \frac{9\pi}{12}, -\frac{\pi}{2} - \frac{11\pi}{12}\right).$$

(b) nontrivial: non-QES levels:

trick: $\mathcal{PT}-$symmetric transformations changing the contour

An **"initial"** $\mathcal{PT}-$symmetric model

$$\left[-\frac{d^2}{dx^2} - (ix)^2 + \lambda\, W(ix)\right]\,\psi(x) = E(\lambda)\,\psi(x)$$

with any **sample potential**:

$$W(ix) = \mathbf{\Sigma}\, g_\beta (ix)^\beta$$

exposed to a **change variables**

$$ix = (iy)^\alpha\,, \qquad \psi(x) = y^\varrho\,\varphi(y).$$

in detail:

at $\alpha > 0$ we have

$$i\,dx = i^{\alpha}\alpha y^{\alpha-1}\,dy, \qquad \frac{(iy)^{1-\alpha}}{\alpha}\frac{d}{dy} = \frac{d}{dx}\,.$$

Gives the equivalent, "Sturmian" problem

(cf. Shanley, PHHQP VI)

i.e., an **“intermediate”** differential equation

$$y^{1-\alpha}\frac{d}{dy}y^{1-\alpha}\frac{d}{dy}\,y^{\varrho}\,\varphi(y)+$$

$$+i^{2\alpha}\alpha^2\left[-(iy)^{2\alpha}+\lambda\,W[(iy)^{\alpha}]-\right.$$

$$\left.-E(\lambda)\right]\,y^{\varrho}\,\varphi(y)=0\,.$$

the first term "behaves",

$$y^{1-\alpha}\frac{d}{dy}y^{1-\alpha}\frac{d}{dy}\,y^{[(\alpha-1)/2]}\,\varphi(y) =$$

$$= y^{2+\varrho-2\alpha}\frac{d^2}{dy^2}\,\varphi(y)+\varrho(\varrho-\alpha)y^{\varrho-2\alpha}\,\varphi(y)\,,$$

at the specific

$$\varrho = \frac{\alpha-1}{2}\,.$$

Conclusion: the new equation

is of **the same** Schrödinger form:

$$-\frac{d^2}{dy^2}\varphi(y) + \frac{\alpha^2 - 1}{4y^2}\varphi(y) +$$

$$+(iy)^{2\alpha-2}\alpha^2\Big[-(iy)^{2\alpha} + \lambda W[(iy)^{\alpha}]\varphi(y) =$$

$$= (iy)^{2\alpha-2}\alpha^2 E(\lambda)\varphi(y)\,.$$

Important: the change of variables

changes the angle between asymptotes

and, hence, it can

diminish the winding number N

Example: polynomial potentials

are interrelated, $\alpha = 1/2$ giving $V_g(y)$ from

$$-\frac{d^2}{dx^2}\,\varphi(x)+\frac{\ell(\ell+1)}{x^2}\,\varphi(x)+V_f(x)\,\varphi(x) = E\,\varphi(x)\,,$$

$$V_f(x) = x^6 + f_4\,x^4 + f_2\,x^2 + f_{-2}\,x^{-2},$$

$$V_g(y) = -(iy)^2+i\,g_1\,y+g_{-1}\,(iy)^{-1}+g_{-2}\,(iy)^{-2}.$$

$\Longrightarrow$ upper sextic $\equiv$ rectified QT HO (pto)

•

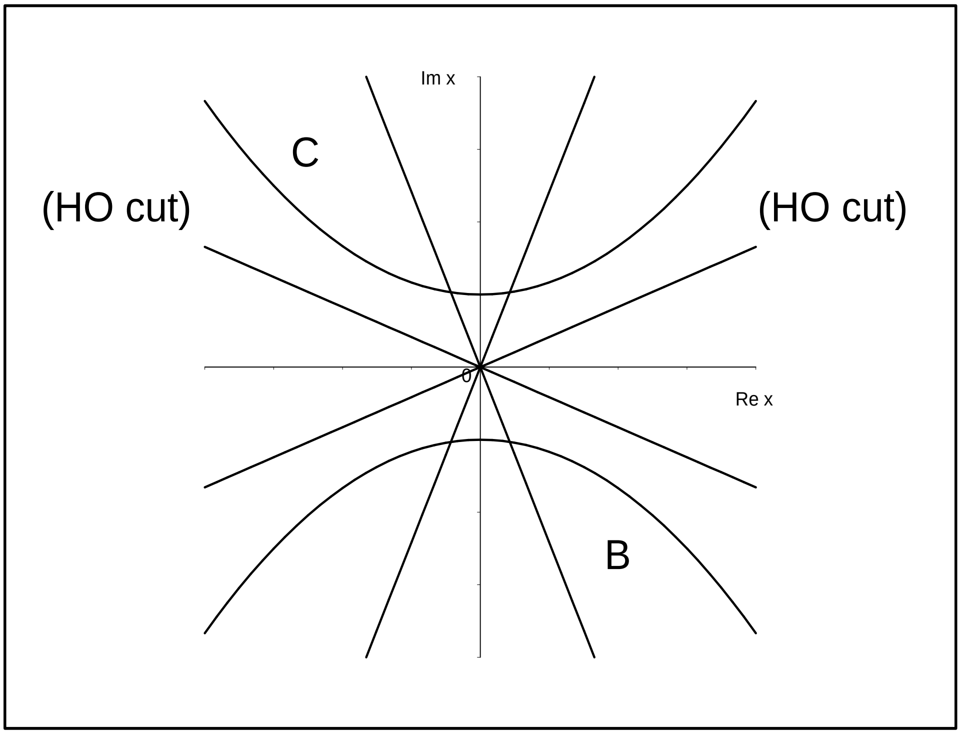

Figure 5: Sextic oscillators B (usual) and C (mapped on HO toboggan)

(c) feasible and useful:

perturbed harmonic oscillators living on a complex curve:

MZ (quant-ph/0606166v1):

Spiked harmonic quantum toboggans

Perturbed harmonic oscillator

$$V(x) = x^2 + \sum_\beta g_{(\beta)}\, x^\beta$$

can be **topologically nontrivial**. Its

$$\psi(x) \approx \psi^{(\pm)}(x) = e^{\pm x^2/2}\,, \qquad |x| \gg 1$$

= multivalued analytic functions

At any $k \in \mathbb{Z}$ they are

(a) "physical" (along a ray $x_\theta = \varrho e^{i\theta}$)

(b) "unphysical". E.g.,

$$\psi^{(-)}(x) = \begin{cases} \psi^{(phys)}(x), & k\pi + \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right), \\ \psi^{(unphys)}(x), & k\pi + \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right). \end{cases}$$

alternatively,

$$\psi^{(+)}(x) = \begin{cases} \psi^{(unphys)}(x), & k\pi + \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right), \\ \psi^{(phys)}(x), & k\pi + \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right). \end{cases}$$

For toboggans we define

$k_f = 0$ and $k_i = 1$ at $N = 0$,

$k_f = -1$ and $k_i = 2$ at $N = 1$,

$k_f = -2$ and $k_i = 3$ at $N = 2$ etc.

Riemann-surface "tobogganic trajectories"

$$\mathcal{D}_{(\varepsilon,N)}^{(PTSQM,\,tobogganic)} = \left\{ x = \varepsilon\,\varrho(\varphi, N)\,e^{i\,\varphi} \right\}$$

$$\varphi \in (-(N+1)\pi,\ N\pi)$$

$$\varrho(\varphi, N) = \sqrt{1 + \tan^2 \frac{\varphi + \pi/2}{2N+1}}$$

$\mathcal{PT}-$symmetry in the presence of the single branch point

.

parity-like operators $\mathcal{P}^{(\pm)} : x \to x{\cdot}\exp(\pm i\pi)$

map $\mathcal{K}_n$ into sheets $\mathcal{K}_{n\pm 1}$.

two eligible rotation-type innovations $\mathcal{T}^{(\pm)}$

same for $\mathcal{P}^{(\pm)}\mathcal{T}^{(\pm)}$ and

$$\left(\mathcal{C}^{(N)}\right)^{\dagger} = \mathcal{D}_{(\varepsilon',N)}^{(PTSQM,\,tobogganic)}\,, \qquad \varepsilon' = \varepsilon\cdot e^{\pm i\pi}\,.$$

Bound states

$$H_{(\mathcal{PT})}\,\psi(x) = E\,\psi(x)$$

with Dirichlet inside the wedges,

$$\psi\left(\varrho \cdot e^{i\,\theta}\right) = 0, \qquad \varrho \gg 1 \quad \theta + k_{i,f}\,\pi \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

spectra = real in unbroken cases.

IV. SCATTERING ALONG THE TOBOGGANS

once "in" and "out" wedge boundaries are

$$\mathcal{A}^{(N)}_{(L)} \to \varrho\, e^{i\,\theta_{in}}, \qquad \theta_{in} = -(N + 3/4)\,\pi,$$

$$\mathcal{A}^{(N)}_{(L)} \to \varrho\, e^{i\,\theta_{out}}, \qquad \theta_{out} = (N - 1/4)\,\pi,$$

$$\mathcal{A}^{(N)}_{(U)} \to \varrho\, e^{i\,\theta_{in}}, \qquad \theta_{in} = -(N + 5/4)\,\pi,$$

$$\mathcal{A}^{(N)}_{(U)} \to \varrho\, e^{i\,\theta_{out}}, \qquad \theta_{out} = (N + 1/4)\,\pi\,.$$

independent solutions become equally large

and oscillate

not only when $V(x) < E$ at $\varrho \to \infty$

but also for many other potentials

including our x^2-dominated sample model.

incoming-beam normalization

$$\psi\left(\varrho\cdot e^{i\,\theta_{in}}\right)=\psi_{(i)}(x)+B\,\psi_{(r)}(x),\quad \varrho\gg 1,$$

and outcoming-beam normalization,

$$\psi\left(\varrho\cdot e^{i\,\theta_{out}}\right)=(1+F)\,\psi_{(t)}(x),\qquad \varrho\gg 1,$$

with incident and reflected waves

$$\psi_{(i,r)}(x)\approx e^{\pm i\varrho^2/2}.$$

$$\left[-\frac{d^2}{dx^2} + \frac{\alpha^2 - 1/4}{x^2} + x^2\right] \psi(x) = E\,\psi(x),$$

set $x^2 = -ir$ along the first nontrivial scattering path $\mathcal{A}^{(0)}_{(L)}$.

"in" branch with $r \ll -1$ and

"out" branch with $r \gg +1$

$$\chi_{(\alpha)}(r) = r^{\frac{1}{4}+\frac{\alpha}{2}}\, e^{ir/2}\, {}_1F_1\left(\frac{\alpha+1-\mu}{2}, \alpha+1; -ir\right),$$

linearly independent partner

$\chi_{(-\alpha)}(r)$, $\alpha \neq n \in I\!N$, $\qquad E = 2\mu.$

$|r| \gg 1$ estimate,

$$r^{\frac{1}{4}+\frac{\alpha}{2}}\chi_{(\alpha)}(r) \approx\ e^{ir/2}\,\frac{r^{\mu/2}\ \exp\left[-i\,\pi\ (\alpha+1)\,/4\right]}{\Gamma\left[(\alpha+1+\mu)/2\right]}+$$

$$+e^{-ir/2}\,\frac{r^{-\mu/2}\ \exp\left[+i\,\pi\ (\alpha+1)\,/4\right]}{\Gamma\left[(\alpha+1-\mu)/2\right]}\,.$$

"rigid" at $\alpha > 0$, $\mu = E/2 > 0$ and

$|x| = |\sqrt(r)| \gg 1$

Note that $\psi_{out}^{(Coul)}(r)$ becomes "distorted",

$$\sin(\kappa r{+}const) \to \sin(\kappa r{+}const{\cdot}\log r{+}const)\,.$$

Similar here, for $\psi_{in,out}(x) \approx$

$$r^{-1/4+(\alpha+\mu)/2}\, e^{ir/2}\, \frac{\exp\left[-i\,\pi\,\left(-\alpha+1\right)/4\right]}{\Gamma\left[(-\alpha+1+\mu)/2\right]} + \ldots\,.$$

V. TOBOGGANS IN POTENTIALS WITH MORE SPIKES

choose **two branch points** $x = \pm 1$,

$$V(x) = V_{regular}(x) + \frac{G}{(x-1)^2} + \frac{G^*}{(x+1)^2}$$

(cf. Sinha A and Roy P 2004 Czechosl. J. Phys. 54 129)

$\Longrightarrow$ **plus:** particle moving along a $\mathcal{PT}-$symmetric "toboggan" path.

(a) an enumeration of the paths

$x^{(QT)}(s)$ encircling two branch points by winding

- counterclockwise around $x^{(BP)}_{(-)}$ (letter L),
- counterclockwise around $x^{(BP)}_{(+)}$ (letter R),
- clockwise around $x^{(BP)}_{(-)}$ ($Q = L^{-1}$),
- clockwise around $x^{(BP)}_{(+)}$ ($P = R^{-1}$).

four-letter alphabet,

$x = x^{(\varrho)}(s)$,

a word ϱ of length $2N$,

$\varrho = \emptyset =$ non-tobogganic

$\mathcal{PT}-$symmetry $L \leftrightarrow R$, $\varrho = \Omega \bigcup \Omega^T$

$\boxed{\text{at } N = 1}$, $\exists$ four possibilities,

$$\Omega \in \left\{ L\,, L^{-1}\,, R\,, R^{-1} \right\}\,, \qquad N = 1\,,$$

$$\varrho \in \left\{ LR\,, L^{-1}R^{-1}\,, RL\,, R^{-1}L^{-1} \right\}\,, \quad N = 1\,.$$

dozen cases $\boxed{\text{at } N=2,}$

$$\Big\{ LL, LR, RL, RR, L^{-1}R, R^{-1}L, LR^{-1},$$

$$RL^{-1}, L^{-1}L^{-1}, L^{-1}R^{-1}, R^{-1}L^{-1}, R^{-1}R^{-1} \Big\}$$

("shorter" $LL^{-1}, L^{-1}L, RR^{-1}, R^{-1}R$

not allowed among $4^2 = 16$ eligible)

$\boxed{\text{at } N = 3,}$ total number $= 36$:

cross 28 out of $4^3 = 64$ words,

$\Omega^{(NA)} = \Omega^{(NAL)} \bigcup \Omega^{(NAR)}$ (prev. L, R)

$\Omega^{(NAL)} = \Omega^{(NAL3)} \bigcup \Omega^{(NAL2)}$

one or two inversions in $\Omega^{(NAL3)}$ (six words),

in $\Omega^{(NAL2)}$ add R or R^{-1} (eight words).

$\boxed{\text{at } N=4}$ we have $256 - 76 - 40 = 140$:

14 elements in $\Omega^{(NAL4)}$

24 elements in $\Omega^{(NAL3)}$, $L \leftrightarrow R$

$\Omega^{(NAL21)}$ (single inversion, 16 elements),

$\Omega^{(NAL22)}$ (two inversions, 8 elements)

$\Omega^{(NAL23)}$ (three inversions, 16 elements).

(b) rectifiable contours $x^{(\varrho_0)}(s)$

recollect: $\mathrm{i}\,x = (\mathrm{i}\,z)^2\,,\ \psi_n(x) = \sqrt{z}\,\varphi_n(z)$

a *strict equivalence* of HO QT to

$$\left(-\frac{d^2}{dz^2} + 4z^6 + 4E_n z^2 + \frac{4\alpha^2-1/4}{z^2}\right)\varphi(z) \,=$$

0 along a *manifestly non-tobogganic* path.

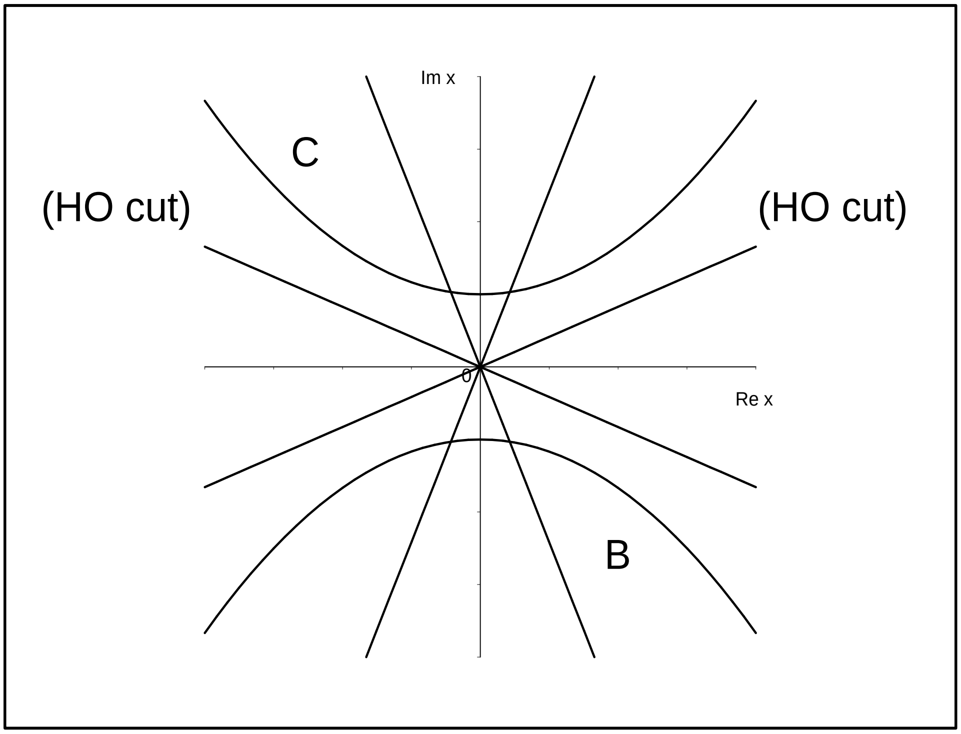

Figure 6: Both the HO-cut lines move upwards, contour C becomes tobogganic

now, to

$$\left[-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{(x-1)^2} + \frac{\ell(\ell+1)}{(x+1)^2} + V(ix)\right]\,\psi(x) =$$

$$= E\,\psi(x)\,.$$

we, similarly, assign the rectified partner

$$\left[-\frac{d^2}{dz^2} + U_{eff}(\mathrm{i}\,z)\right]\,\varphi(z) = 0$$

$$U_{eff}(\mathrm{i}\,z) = U(\mathrm{i}\,z) + \frac{\mu(\mu+1)}{(z-1)^2} + \frac{\mu(\mu+1)}{(z+1)^2} \equiv$$

$$\equiv U(\mathrm{i}\,z) + 2\,\frac{\mu(\mu+1)[1-(\mathrm{i}\,z)^2]}{\left[1+(\mathrm{i}\,z)^2\right]^2}.$$

implicit rectification formula

$$1 + (ix)^2 = \left[1 + (iz)^2\right]^{\kappa}, \qquad \kappa > 1$$

$z = -\mathrm{i}\,\varrho$ on itself:

explicit rectification formula

$$x = -\mathrm{i}\,\sqrt{(1 - z^2)^{\kappa} - 1}$$

Effective non-tobogganic potentials - construction = routine:

$$\frac{d}{d\,x} = \beta(z)\,\frac{d}{d\,z}\,, \qquad \beta(z) = -\mathrm{i}\,\frac{\sqrt{(1-z^2)^\kappa - 1}}{\kappa\,z\,(1-z^2)^{\kappa-1}}\,.$$

$$\psi(x) = \chi(z)\,\varphi(z)\,\text{with}\,\chi(z) = const\,/\sqrt{\beta(z)}$$

(Liouville L 1837 J.Math.Pures Appl. 1 16)

$$\left(-\beta(z)\,\frac{d}{dz}\,\beta(z)\,\frac{d}{dz}+V_{eff}[ix(z)]-E\right)\,\chi(z)\,\varphi(z)=$$

$$U_{eff}(\mathrm{i}\,z)=\frac{V_{eff}[\mathrm{i}\,x(z)]-E_n}{\beta^2(z)}+\frac{\beta''(z)}{2\,\beta(z)}-\frac{[\beta'(z)]^2}{4\,\beta^2(z)}$$

QED

shapes of the tobogganic pull-backs:

the vicinity of the negative imaginary axis

$$z=-\mathrm{i}\,r\,e^{\mathrm{i}\,\theta}\ \longrightarrow\ x=-\mathrm{i}\,\left[\left(1+r^2\,e^{2\,\mathrm{i}\,\theta}\right)^{\kappa}-1\right]^{1/2}\,.$$

factor $\sqrt{\kappa}$ at the small radii r

parallelism at $r \gg 1$

consequences:

knot-like $x^{\varrho_0}(s)$ by computer graphics,

straight-line $z(s) = s - \mathrm{i}\,\varepsilon$ pulled back

N sensitive to ε (pto)

.

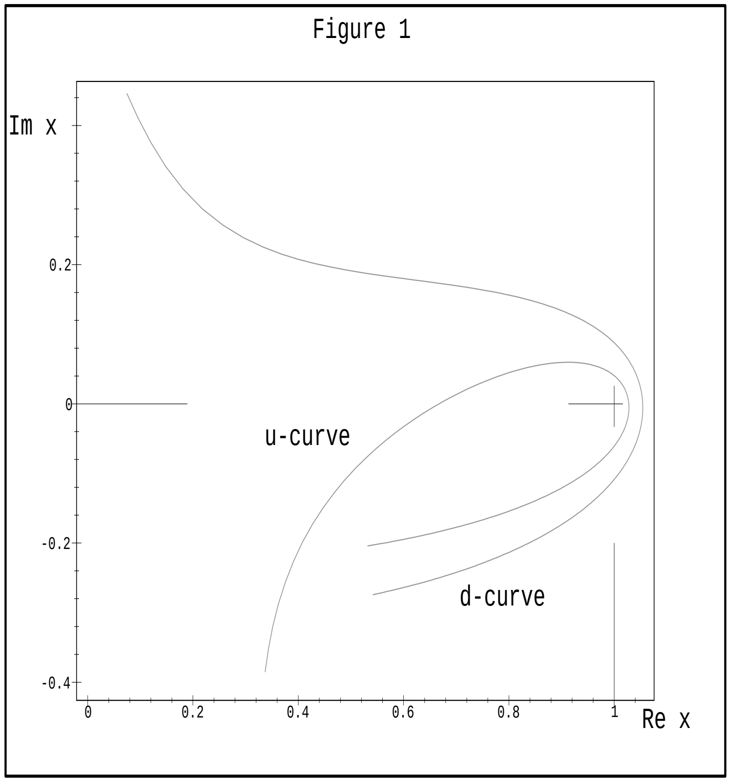

Figure 7: Two bitoboggans ($\kappa = 2.4$, $s \in (0.4, 1.4)$)

favorable property:

winding number grows *quickly* with κ

(pto)

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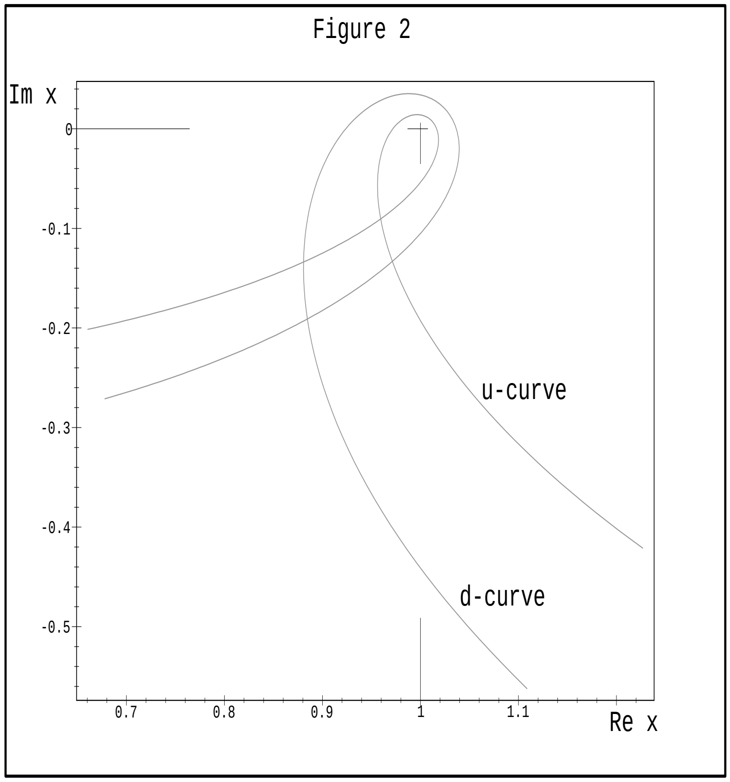

Figure 8: Two bitoboggans ($\kappa = 3$, $s \in (0.4, 1.4)$)

user-friendliness:

winding numbers *arbitrarily* large

paths *very close* to BPs

the sensitivity to the shift *recurs*

(pto)

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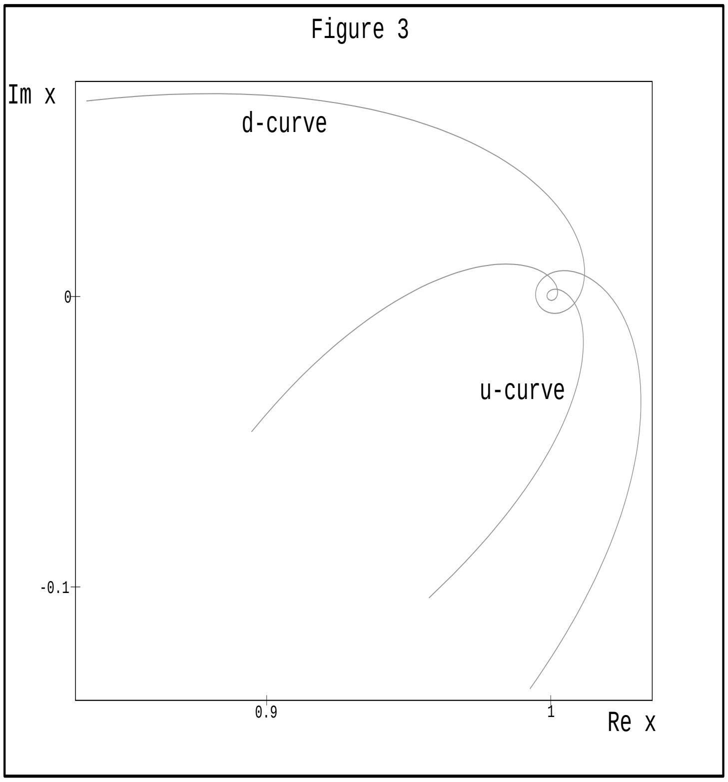

Figure 9: Two bitoboggans ($\kappa = 5$, $s \in (0.4, 1.4)$)

III. OUTLOOK:

Three-Hilbert-space formulation of QM

1. relativistic QM in $\mathcal{H}^{(2,3)}$:

= first-quantized Klein-Gordon [M '03]

$$H = \begin{pmatrix} 0 & -\triangle + m^2(x) \\ I & 0 \end{pmatrix}$$

= channel-coupling models [Z '06]

= first-quantized Proca [JS '06, SJZ '07]

2. beyond QM:

$=\alpha^2$ **dynamos in MHD [GK '06, GZ '07]**

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Complex spectra admitted in physical regime.

New mathematics needed for perturbations in boundary conditions.

= cosmology [M ’03, ACK ’06]

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VI. SUMMARY

- in rectified tobogganic contours $x^{(\varrho_0)}(s)$

  descriptor word ϱ_0 inferred *a posteriori*

- QT2 observable *if and only if*

  $\mathcal{PT}-$symmetry unbroken

- topology-dependent spectra