30 years of many-body in Spain: looking into the future

of mathematical methods in many-body physics

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Spiked A-body anharmonic oscillators in a strong-repulsion perturbative picture

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I. General subject: $A-$particle bound-state problems

$$\hat{H}\Psi(\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_A) = E\,\Psi(\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_A)\,.$$

Here, their first nontrivial case with $A = 3$ and $x_j \in I\!R$,

$$\left[-\sum_{i=1}^{3}\frac{\partial^2}{\partial {x_i}^2}+\hat{W}\right]\Psi = E\,\Psi\,,$$

$$\hat{W} = V(x_1-x_2)+V(x_2-x_3)+V(x_3-x_1)\,.$$

The best known Calogero's solvable option

$$V^{(spiked)}(x_i - x_j) = \frac{\nu(\nu+1)}{(x_i - x_j)^2} + \omega^2 (x_i - x_j)^2$$

Here, its $L, K = 1, 2, \dots$ generalizations

$$V(x_i - x_j) = \alpha^2 (x_i - x_j)^L + \beta^2 (x_i - x_j)^{-K}$$

and their superpositions.

II. The method

Taylor identities in the strong-repulsion regime:

(1) SHO has a minimum at $R = [\nu(\nu+1)/\omega^2]^{1/4}$**,**

$$V_{eff}(r) = \frac{\nu(\nu+1)}{r^2} + \omega^2 r^2$$

$$= V_{eff}(R) + \frac{1}{2}(r-R)^2\, V''_{eff}(R) + \mathcal{O}\left[(r-R)^3\right]$$

$$= 2\omega^2 R^2 + 4\omega^2 (r-R)^2 + \mathcal{O}\left[\frac{(r-R)^3}{R}\right],$$

$$= \text{solv. HO well at } 1/R \approx 1/\sqrt{\nu} \ll 1.$$

(2) Taylor identities in the quartic case with $r = R + x$

clarify the determination of R as a root of $\mathcal{O}(x)$ term, etc.,

$$V_{eff}(r) = \left[\frac{\nu\ (\nu+1)}{R^2} + \omega^2 R^2 + \lambda\, R^4\right] +$$
$$+2\, \left[\omega^2 R - \frac{\nu\ (\nu+1)}{R^3} + 2\,\lambda\, R^3\right] x +$$
$$+ \left[\omega^2 + 3\,\frac{\nu\ (\nu+1)}{R^4} + 6\,\lambda\, R^2\right] x^2 +$$
$$+4\, \left[-\frac{\nu\ (\nu+1)}{R^5} + \lambda\, R\right] x^3 + O\left(x^4\right).$$

(3) An unsolvable example of UE $\longleftrightarrow$ RHO $\longleftrightarrow$ SHO**,**

$$W(r) = F\,r^3 + \frac{G}{r^3}, \qquad r,\ F,\ G > 0\,.$$

$$G \gg 1,\ r_{min} = R = (G/F)^{1/6} \gg 1$$

$$W(R+\xi) = 2\,\sqrt{G\,F} + 9\,\sqrt{G\,F}\,\frac{\xi^2}{R^2} - \ldots\,.$$

$$W_0(R_0+\xi) = F_0\,(R_0+\xi)^2 + \frac{G_0}{(R_0+\xi)^2}$$

$$F_0 = \frac{9}{4}\,\sqrt[6]{G\,F^5}, \qquad G_0 = \frac{4}{9}\,\sqrt[6]{G^5\,F}\,.$$

III. The $A = 3$ models in Jacobi coordinates

$$\begin{pmatrix} Z \\ X \\ Y \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} Z \\ X \\ Y \end{pmatrix} .$$

with interactions

$$\hat{W}(x_1, x_2, x_3) = \sum_{m=-K}^{L} F_m\, W^{(m)}(x_1, x_2, x_3),$$

$$F_L = \alpha^2 > 0, \quad F_{-K} = \beta^2 > 0,$$

$$W^{(m)} = (x_1 - x_2)^m + (x_2 - x_3)^m + (x_3 - x_1)^m$$

and Schrödinger equation with $U^{(m)}(X, Y) \equiv W^{(m)}(x_1, x_2, x_3)$,

$$\left\{ -\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} + \sum_{m=-K}^{L} F_m\, U^{(m)} - E \right\} \Phi = 0$$

A polar re-parametrization of the coordinates,

$$X = X(\varrho, \varphi) = \varrho \sin\varphi$$

$$Y = Y(\varrho, \varphi) = \varrho \cos\varphi$$

gives Schrödinger equation with $\Omega^{(m)}(\varrho, \varphi) \equiv U^{(m)}(X, Y)$,

$$\left\{-\frac{\partial^2}{\partial \varrho^2} - \frac{1}{\varrho^2}\frac{\partial^2}{\partial \varphi^2} + \sum_{m=-K}^{L} F_m \Omega^{(m)} - E\right\} \Phi = 0$$

Note the "forgotten" separability for the

1st illustrative oscillator: quartic,

$$U^{(2)}(X,Y) = 3\,X^2 + 3\,Y^2$$

$$\Omega^{(2)}(\varrho,\varphi) = U^{(2)}[X(\varrho,\varphi),Y(\varrho,\varphi)] = 3\,\varrho^2$$

$$U^{(4)}(X,Y) = \frac{9}{2}\left(X^2 + Y^2\right)^2 = \frac{9}{2}\,\varrho^4$$

A guiding return to Calogerian $U^{(SHO)}(X,Y)$ reveals that

(1) its $A=3$ spiked part = a trigonometric identity:

$$\frac{1}{2X^2}+\frac{1}{2(X-\sqrt{3}Y)^2}+\frac{1}{2(X+\sqrt{3}Y)^2}\equiv$$

$$\equiv\frac{1}{2\varrho^2}\left(\frac{1}{\sin^2\varphi}+\frac{1}{\sin^2(\varphi+\frac{2}{3}\pi)}+\frac{1}{\sin^2(\varphi-\frac{2}{3}\pi)}\right)\equiv$$

$$\equiv\frac{9}{2\,\varrho^2\,\sin^2 3\,\varphi}$$

and its Schrödinger equation = also separable.

(2) its angular Taylor is easy,

$$\frac{1}{\sin^2 3(\gamma + \pi/6)} = 1 + 9\,\gamma^2 + 54\,\gamma^4 + \frac{1377}{5}\,\gamma^6 + \ldots$$

while the radial is no news, with $R = \sqrt[4]{\frac{3\nu(\nu+1)}{2\omega^2}}$ **and**

$$V_{eff}(\varrho) = \frac{9\nu(\nu+1)}{2\varrho^2} + 3\,\omega^2\varrho^2 =$$

$$= V_{eff}(R) + \frac{1}{2}(\varrho - R)^2\, V''_{eff}(R) + \ldots =$$

$$= 6\omega^2 R^2 + 12\omega^2(\varrho - R)^2 + \ldots$$

(3) test passed - the approximants prove correct,

$$\Omega^{(SHO)}(R+\xi,\pi/6+\eta/R)\approx$$

$$\approx 6\omega^2 R^2 + 12\omega^2\xi^2 + 27\,\omega^2\,\eta^2$$

with $\mathcal{O}(\eta^m \times \xi^n) \approx 1/R^{m+n-2} \ll 1$ while

$$E - 6\omega^2 R^2 \approx$$

$$2\sqrt{3}\,\omega\,(2n+1) + 3\sqrt{3}\,\omega(2m+1) + \ldots\,.$$

(4) Weyl chambers = cartesian, rotated $\pi/6$;

kinetic energy = *locally* cartesian =

$$= -\frac{\partial^2}{\partial\varrho^2} - \frac{1}{\varrho^2}\frac{\partial^2}{\partial\varphi^2} =$$

$$-\frac{\partial^2}{\partial\varrho^2} - \frac{1}{(R+\xi)^2}\frac{\partial^2}{\partial\varphi^2} = -\frac{\partial^2}{\partial\xi^2} - \left[1 + \mathcal{O}\left(\frac{\xi}{R}\right)\right]\frac{\partial^2}{\partial\eta^2}$$

to be verified for $V = \omega^2\ (x_i - x_j)^2 + \gamma\ (x_i - x_j)^3 + \frac{\nu(\nu+1)}{(x_i-x_j)^2}$.

IV. Choose $L = 3$

1. fix $x_1 > x_2 > x_3$, i.e., $\varphi \in (0, \pi/3)$
2. deduce $\Omega^{(3)}(\varrho, \varphi) \geq 0$ for $\gamma > 0$
3. study the extreme $\omega = 0$
4. spectrum = discrete since two minima

at $\varphi_\pm = \pi/6 - \varphi_\mp$ in

$$\Omega^{(SC)}(\varrho, \varphi) = \alpha^2 \, \varrho^3 \sin 3\varphi + \frac{\beta^2}{\varrho^2 \sin^2 3\varphi}$$

with

$$\alpha^2 = \frac{3}{2}\gamma > 0, \quad \beta^2 = \frac{9}{2}\nu(\nu+1) > 0$$

and with the $\varrho-$dependence rule

$\partial_\varphi \Omega^{(SC)} = 0$ and its form

$$\sin^2 3\varphi_\pm = \frac{2\beta^2}{\alpha^2\varrho^5}$$

implying an asymptotic growth,

$$\Omega^{(SC)} > \frac{3\varrho}{2}\left(2\alpha^4\beta^2\varrho\right)^{1/3}, \quad \varrho \geq \varrho_0 = \left(\frac{2\beta^2}{\alpha^2}\right)^{1/5}.$$

(5) general M in $\Omega^{(2M+1)}(\varrho, \varphi)$ at $\varrho \gg 1$:

$$\Omega = \alpha^2 \, \Omega^{(2M+1)} + \beta^2 \, \Omega^{(-2)} \sim \varrho^{4M/3}, \qquad .$$

is growing quickly at the small

$$\varphi_- = \pi/3 - \varphi_+ \sim \varrho^{-(2M+3)/3}.$$

(6) specific $K = 3$

two minima at $\varrho \gg 1$ and the three valleys merge at

$$\varrho = \varrho_0 = \left(2\beta^2/\alpha^2\right)^{1/5}$$

absolute minimum at $\varphi = \varphi_0 = \pi/6$ and

$$R = \varrho_0^{(min)} = \sqrt[5]{2\beta^2/(3\alpha^2)} \gg 1$$

Re-define $\beta^2 = 3\,\alpha^2\,R^5/2$, $\varrho = R + \xi$,

$\varphi = \pi/6 + \eta/R$ and expand

$$\frac{1}{\alpha^2}\,\Omega^{(SCO)}(\varrho,\varphi) = \varrho^3 \sin 3\varphi + \frac{\beta^2}{\alpha^2\,\varrho^2 \sin^2 3\varphi} =$$

$$= \frac{5}{2}R^3 + 9\,R\eta^2 + \frac{15}{2}\,R\xi^2 -$$

$$-\frac{81}{2}\,\xi\,\eta^2 - 5\,\xi^3 + \frac{1}{R}\,\left(\frac{675}{8}\,\eta^4 + 27\,\xi^2\eta^2 + \frac{15}{2}\,\xi^4\right)$$

$$+\mathcal{O}\left(\frac{1}{R^2}\right)\,.$$

Schrödinger equation

$$\left[-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \alpha^2 \left(\frac{15}{2}\, x^2 + 9\, y^2\right) + \mathcal{O}\left(\sigma^5\right)\right] \psi =$$

$$= \sigma^2 \left(E - \frac{5}{2} R^3\right) \psi$$

with $\sigma = 1/\sqrt[4]{R}$, $\varrho = R + \sigma\, x$ and

$$\varphi = \pi/6 + \sigma\, y/R.$$

Leading-order energies: closed formula

$$E = E_{m,n} = \frac{5}{2}\,R^3 +$$

$$+\alpha\sqrt{\frac{15\,R}{2}}\,(2m+1) + 3\,\alpha\,\sqrt{R}\,(2n+1) + \mathcal{O}\left(\frac{1}{R^{3/4}}\right) \,.$$

V. Potentials $U^{(m)}$ with even $m = 2M$

$$\text{Complicated } U^{(6)}(X,Y) =$$

$$\frac{33}{4}\,X^6 + \frac{45}{4}\,X^4Y^2 + \frac{135}{4}\,X^2Y^4 + \frac{27}{4}\,Y^6,$$

$$U^{(8)}(X,Y) = \frac{3}{8}\left(X^2 + Y^2\right)\times$$

$$\times\left(27\,Y^6 + 225\,X^2Y^4 - 15\,X^4Y^2 + 43\,X^6\right),$$

abbreviation

$$\Omega^{(2M)}(\varrho,\varphi) = U^{(2M)}[X(\varrho,\varphi), Y(\varrho,\varphi)]$$

compact formulae

$$\Omega^{(6)}(\varrho,\varphi) = \frac{3\,\varrho^6}{4}\left(9 + 2\,\sin^2 3\varphi\right),$$

$$\Omega^{(8)}(\varrho,\varphi) = \frac{3\,\varrho^8}{8}\,\left(27 + 16\,\sin^2 3\varphi\right)\,,$$

$$\Omega^{(10)}(\varrho,\varphi) = \frac{27\,\varrho^{10}}{16}\,\left(9 + 10\,\sin^2 3\varphi\right)\,,$$

$$\Omega^{(12)}(\varrho,\varphi) =$$

$$\frac{729\,\varrho^{12}}{32} + \frac{81\,\varrho^{12}}{2}\,\sin^2 3\,\varphi + \frac{3\,\varrho^{12}}{4}\,\sin^4 3\,\varphi\,.$$

In general, $\frac{\Omega^{(2M)}(\varrho,\varphi)}{\varrho^{2M}} =$

$$\frac{3^M}{2^{M-1}} + c_1\ \sin^2 3\varphi + \ldots + c_N\ \sin^{2N} 3\varphi$$

where, formally,

$$N = entier\left[\frac{M}{3}\right], \qquad M = 1, 2, \ldots$$

and $entier[x]$ denotes the integer part of a real number x.

VI. Potentials $\Omega^{(m)}$ with odd $m = 2M + 1$

$$U^{(3)}(X,Y) = \frac{3}{2}\sqrt{2}\,X\left(X^2 - 3\,Y^2\right)\,,$$

$$U^{(5)}(X,Y) = \frac{15}{4}\sqrt{2}\left(X^2 - 3\,Y^2\right)\left(X^2 + Y^2\right)X,$$

$$U^{(7)}(X,Y) = \frac{63}{8}\sqrt{2}\left(X^2 - 3\,Y^2\right)X\left(X^2 + Y^2\right)^2,$$

$$U^{(9)}(X,Y) = X\,\left(X^2 - 3\,Y^2\right)\times$$

$$\times\frac{3}{16}\sqrt{2}\left(81\,Y^6 + 279\,X^2Y^4 + 219\,X^4Y^2 + 85\,X^6\right)$$

while, in polar coordinates,

$$\Omega^{(3)}(\varrho,\varphi) = \frac{3\,\varrho^3}{2}\,\sqrt{2}\,\sin 3\varphi,$$

$$\Omega^{(5)}(\varrho,\varphi) = \frac{15\,\varrho^5}{4}\,\sqrt{2}\,\sin 3\varphi,$$

$$\Omega^{(7)}(\varrho,\varphi) = \frac{63\,\varrho^7}{8}\,\sqrt{2}\,\sin 3\varphi,$$

$$\Omega^{(9)}(\varrho,\varphi) = \frac{3\,\varrho^9}{16}\,\sqrt{2}\,\sin 3\varphi\,\left(81 + 4\,\sin^2 3\varphi\right) =$$

$$= \frac{3\,\varrho^9}{16}\,\sqrt{2}\,\sin 3\varphi\,\left(83 - 2\,\cos\,6\,\varphi\right),$$

$$\Omega^{(11)}(\varrho,\varphi) = \frac{33\,\varrho^{11}}{32}\,\sqrt{2}\,\sin 3\varphi\,\left(27 + 4\,\sin^2 3\varphi\right)\,,$$

$$\Omega^{(13)}(\varrho,\varphi) = \frac{117\,\varrho^{13}}{64}\,\sqrt{2}\,\sin 3\varphi\,\left(27 + 8\,\sin^2 3\varphi\right)\,,$$

etc. Again, $m = 6N + 3$ or $m = 6N + 5$ or

$m = 6N + 7$ leads to the general rule

$$\frac{\Omega^{(2M+1)}(\varrho,\varphi)}{\varrho^{2M+1}} =$$

$$= (2M+1)\,\sqrt{2}\,\sin\,3\varphi\times$$

$$\times\left(\frac{3^{M-1}}{2^M} + d_1\;\sin^2 3\varphi + \ldots + d_N\;\sin^{2N} 3\varphi\right)$$

where

$$N = entier\left[\frac{M-1}{3}\right]\,, \qquad M = 1, 2, \ldots\,.$$

VII. Strongly singular potentials $\Omega^{(m)}$ with negative m

$$\Omega^{(-1)}(\varrho,\varphi) = -\frac{3}{\varrho\,\sqrt{2}\,\sin 3\varphi},$$

$$\Omega^{(-2)}(\varrho,\varphi) = \left(\frac{3}{\varrho\,\sqrt{2}\,\sin 3\varphi}\right)^2 = \left[\Omega^{(-1)}(\varrho,\varphi)\right]^2.$$

$$\Omega^{(-3)}(\varrho,\varphi) = \left[\Omega^{(-1)}(\varrho,\varphi)\right]^3 \left(1 - \frac{4}{9}\,\sin^2\,3\,\varphi\right),$$

$$\Omega^{(-4)}(\varrho,\varphi) = \left[\Omega^{(-1)}(\varrho,\varphi)\right]^4 \left(1 - \frac{16}{27}\,\sin^2\,3\,\varphi\right),$$

$$\Omega^{(-5)}(\varrho,\varphi) = \left[\Omega^{(-1)}(\varrho,\varphi)\right]^5 \left(1 - \frac{20}{27}\,\sin^2\,3\,\varphi\right),$$

$$\Omega^{(-6)}(\varrho,\varphi) = \left[\Omega^{(-1)}(\varrho,\varphi)\right]^6 \times$$
$$\times \left(1 - \frac{904}{667}\,\sin^2\,3\,\varphi + \frac{8248}{18009}\,\sin^4 3\,\varphi\right).$$

VIII. SUMMARY:

(A) Why are the spiked oscillators interesting?

(1) the old Aguilera's and Guardiola's answer in JMP '91:

$$E^{(ground\ state)}_{(spiked\ HO)} \text{ at } K = 5/2$$

[= the E. M. Harrell's Ann. Phys. '77 $A = D = 1$ **formula]**

has the same structure as the many-bosonic energy

$$E = a + b\,\beta^2 + c\,\beta^4 \log\beta + \mathcal{O}(\beta^6)$$

(2) they offer insights via analytic continuation within PT symmetric quantum mechanics:

(*) single-particle case: FGRZ, JPA '98

(*) three-particle case: ZT, JPA '02

(3) they may exhibit partial solvability, etc.

(4) they have immediate impact on perturbation methods: BFZ PRA '89, AEG JMP '90, FGZ CJP '91

(*) facilitated Padé-type resummations: FGZ PRA '93

(*) feasible $\beta \gg 1$ expansions in the past (GSR NC '92, $A = 1$), today ($A = 3$) and in the future ($A > 3$)

(B) What's shared by $A = 1, 2$ and $A = 3$?

(Σ 1) mathematics:

(*) a Lie-algebraic background (Calogero)

(Σ 2) computing methods:

(*) a rigorous treatment of divergent series

(*) feasibility of symbolic manipulations: ZYG JPA '03

(Σ 3) guidance towards $A > 3$

(*) expansions using artificial parameters: Z PRA '87 and

M. Znojil, *Low-lying spectra ...*", **JPA 36 (2004) 9929-41**