.

Toboggans in Quantum Mechanics: a new family of models

Miloslav Znojil

NPI AS CR, 250 68 Řež, Czech Republic

QTS-5, Valladolid, 23. VII. 2007

PLAN OF THE TALK

- **I.** INTRODUCTION: $H \neq H^\dagger = \eta\, H\, \eta^{-1}$, $\eta = \mathcal{P}$

- **II.** THE CONCEPT OF QUANTUM TOBOGGANS

- **III.** QT CONSTRUCTIONS

- **IV.** SCATTERING

- **V.** QT MODELS WITH TWO BRANCH POINTS

- **VI.** SUMMARY: $H^\dagger = \Theta\, H\, \Theta^{-1}$

I. INTRODUCTION:

$\mathcal{PT}$−symmetric quantum mechanics

.

(a) prehistory:

$\exists$ *complex* $V(x)$ *with real spectra:*

sample: Buslaev and Grecchi, 1993:

$V(x) \sim -x^4$ at $|x| \gg 1$:

exhibited $\mathcal{PT}-$symmetry

•

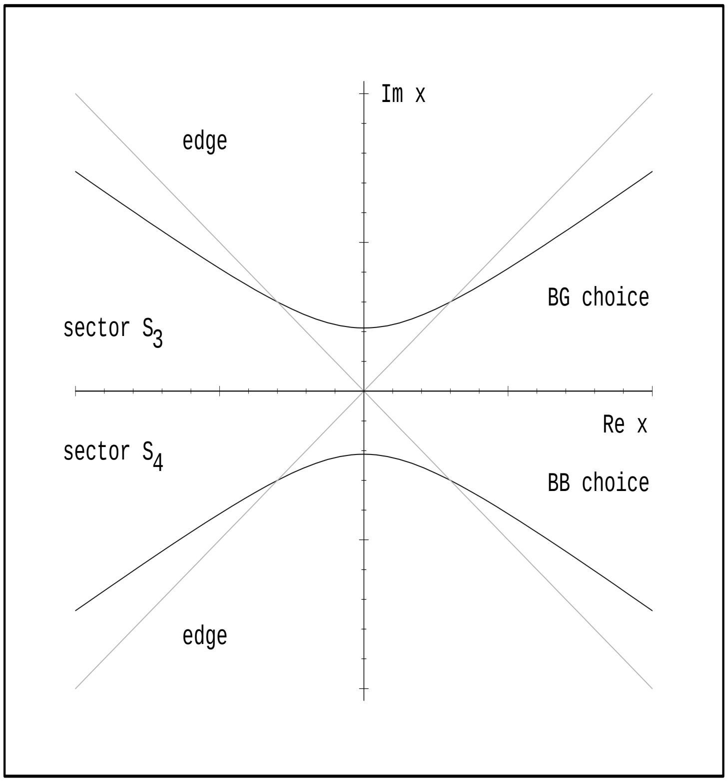

Figure 1: Complex curves of coordinates (BG oscillator)

explanation: $\exists$ a **Hermitian** equivalent of

$$H_{(\mathcal{PT})}\,\psi(x) = E\,\psi(x)$$

with Dirichlet abcs

$$\psi\left(\varrho \cdot e^{i\,\theta}\right) = 0, \qquad \varrho \gg 1$$

inside the wedges where, e.g.,

$$\theta_{left\,down} \in \left(-\frac{3\pi}{3}, -\frac{2\pi}{3}\right)$$

(Smilga: a “cryptoreality” of the spectrum)

(b) 1998 = year zero:

class of BB's PT symmetric potentials:

$$V(x) = V_{symm}(x) + \mathrm{i}\, V_{antisymm}(x)$$

.

(c) 2001 = year one:

DDT's proofs of the reality of the spectra

.

(d) 2005 = the birth of QTs:

- MZ, quant-ph/0502041

  Phys. Lett. A 342 (2005) 36 - 47

- MZ, quant-ph/0606166

  J. Phys. A: Math. Gen. 39 (2006) 13325 - 13336

·

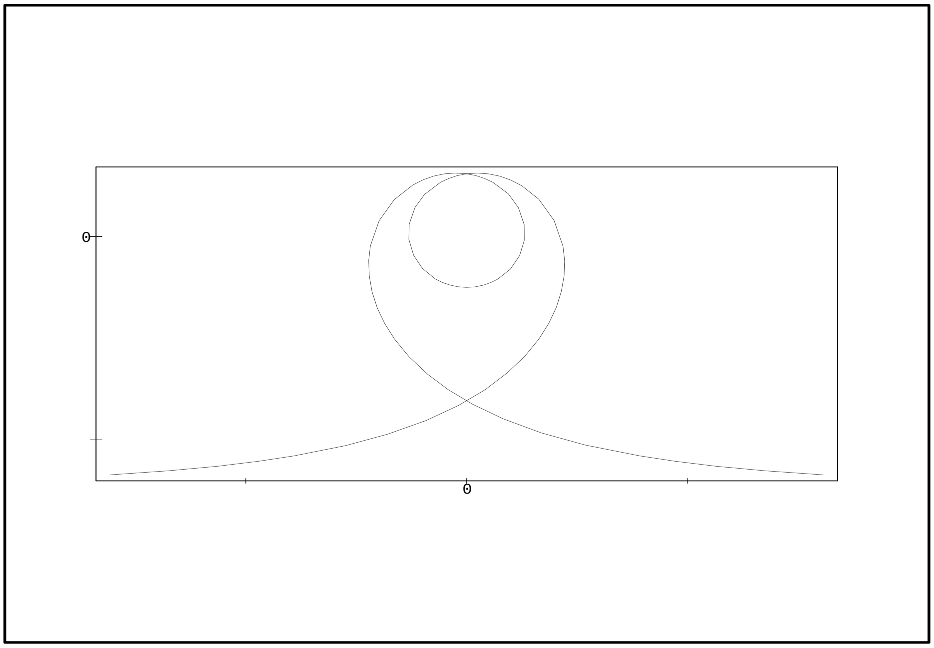

Figure 2: BG-oscillator toboggan at $\ell \neq -1, 0$, with $\mathcal{N} = 2$

II. MODELS ON COMPLEX CONTOURS $\mathcal{C}^{(N)}$

(a) the first step: spiked HOs

MZ,

PT symmetric harmonic oscillators

Phys. Lett. A 259 (1999) 220 - 3.

$$\left(-\frac{d^2}{dx^2}+\frac{\ell(\ell+1)}{x^2}+x^2\right)\psi(x)=E\,\psi(x)$$

defined along straight contour

$$\mathcal{C}^{(0)} = \{x \mid x = t - i\,\varepsilon,\, t \in I\!R\}$$

$\exists$ "twice as many" bound-state levels

$$E = E_{n,\ell,\pm} = 4n + 2 \pm 2\alpha(\ell)$$

$=$ *toboganically trivial*

(b) the second step: AHOs

(b.1) non-tobogganic abcs:

$$\left[-\frac{d^2}{dx^2} + V_{\mathcal{PT}}(x)\right] \psi(x) = E\,\psi(x)$$

$$\psi(\pm\text{Re } L + i\,\text{Im } L) = 0\,,$$

$$|L| \gg 1 \quad \text{or} \quad |L| \to \infty\,.$$

(b.2) tobogganic, along loops

on multisheeted Riemann surfaces

with, say, $\varphi \in (-(N+1)\pi,\ N\pi)$ in

$$\mathcal{C}^{(N)} = \left\{ x = \varepsilon\, \varrho(\varphi, N)\, e^{i\,\varphi}\,, \varepsilon > 0 \right\}\,.$$

$$\varrho(\varphi, N) = \sqrt{1 + \tan^2 \frac{\varphi + \pi/2}{2N+1}}$$

.

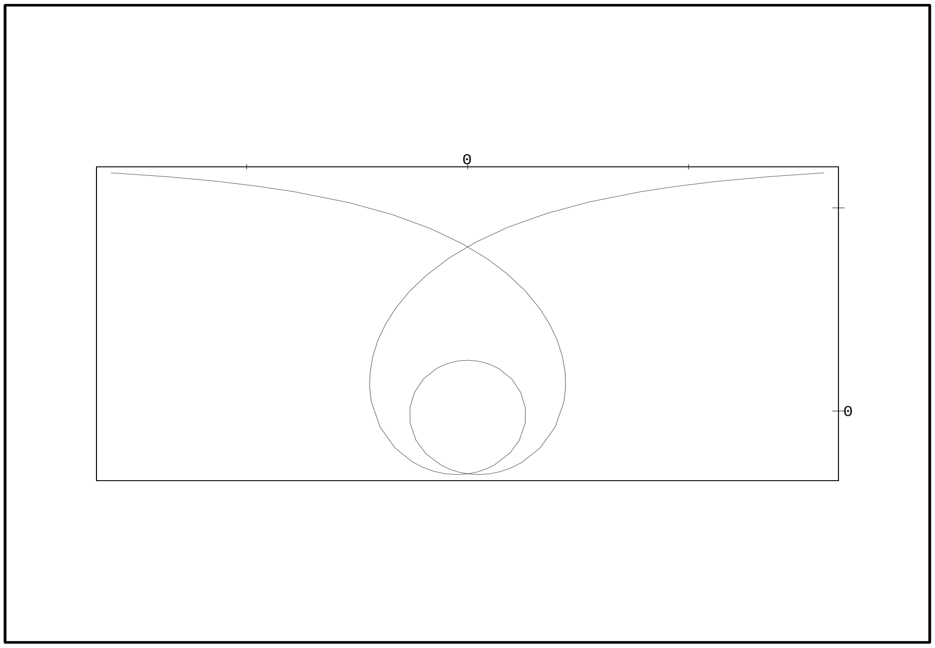

Figure 3: Upside down? Winding number still $\mathcal{N} = 2$

necesssary: branch point in $\psi(x)$, say, from

$$V(x) \sim \frac{irrational\ constant}{x^2}$$

what is then the $\mathcal{PT}-$symmetry of $\psi(x)$?

the left-right symmetry of $\mathcal{C}^{(N)}$

along the whole Riemann surface.

III. CONSTRUCTIONS OF QUANTUM TOBOGGANS

(a) QES models

Miloslav Znojil (quant-ph/0502041):

PT-symmetric quantum toboggans

Phys. Lett. A 342 (2005) 36-47.

model:

$$V(x) = x^{10} + \text{asymptotically smaller terms}$$

$$\psi(x) = e^{-x^6/6 + \text{asymptotically smaller terms}}$$

reparametrized

$$\psi(x) = \exp\left[-\frac{1}{6}\varrho^6 \cos 6\varphi + \ldots\right],$$

$\exists$ five non-tobogganic wedges:

$$\Omega_{(first\ right)} = \left(-\frac{\pi}{2} + \frac{\pi}{12}, -\frac{\pi}{2} + \frac{3\pi}{12}\right),$$

$$\Omega_{(first\ left)} = \left(-\frac{\pi}{2} - \frac{\pi}{12}, -\frac{\pi}{2} - \frac{3\pi}{12}\right),$$

$$\Omega_{(third\ right)} = \left(-\frac{\pi}{2} + \frac{5\pi}{12}, -\frac{\pi}{2} + \frac{7\pi}{12}\right), \quad \ldots$$

$$\ldots \quad \Omega_{(fifth\ left)} = \left(-\frac{\pi}{2} - \frac{9\pi}{12}, -\frac{\pi}{2} - \frac{11\pi}{12}\right).$$

(b) nontrivial: non-QES levels:

trick: $\mathcal{PT}-$symmetric transformations changing the contour

An **"initial"** $\mathcal{PT}-$symmetric model

$$\left[-\frac{d^2}{dx^2} - (ix)^2 + \lambda\, W(ix)\right]\, \psi(x) = E(\lambda)\, \psi(x)$$

with any **sample potential**:

$$W(ix) = \mathbf{\Sigma}\, g_\beta (ix)^\beta$$

exposed to a **change variables**

$$ix = (iy)^\alpha\,, \qquad \psi(x) = y^\varrho\, \varphi(y).$$

in detail:

at $\alpha > 0$ we have

$$i\,dx = i^\alpha \alpha y^{\alpha-1}\,dy, \qquad \frac{(iy)^{1-\alpha}}{\alpha}\frac{d}{dy} = \frac{d}{dx}\,.$$

Gives the equivalent, “Sturmian” problem

(cf. Shanley, PHHQP VI)

i.e., an **“intermediate”** differential

equation

$$y^{1-\alpha}\frac{d}{dy}y^{1-\alpha}\frac{d}{dy}\,y^{\varrho}\,\varphi(y)+$$

$$+i^{2\alpha}\alpha^{2}\Big[-(iy)^{2\alpha}+\lambda\,W[(iy)^{\alpha}]-$$

$$-E(\lambda)]\;y^{\varrho}\,\varphi(y)=0\,.$$

the first term "behaves",

$$y^{1-\alpha}\frac{d}{dy}y^{1-\alpha}\frac{d}{dy}\,y^{[(\alpha-1)/2]}\,\varphi(y) =$$

$$= y^{2+\varrho-2\alpha}\frac{d^2}{dy^2}\,\varphi(y)+\varrho(\varrho-\alpha)y^{\varrho-2\alpha}\,\varphi(y)\,,$$

at the specific

$$\varrho = \frac{\alpha - 1}{2}\,.$$

Conclusion: the new equation

is of **the same** Schrödinger form:

$$-\frac{d^2}{dy^2}\,\varphi(y) + \frac{\alpha^2 - 1}{4y^2}\,\varphi(y) +$$

$$+(iy)^{2\alpha-2}\alpha^2\left[-(iy)^{2\alpha} + \lambda\,W[(iy)^\alpha]\,\varphi(y) =\right.$$

$$= (iy)^{2\alpha-2}\alpha^2\,E(\lambda)\,\varphi(y)\,.$$

Important: the change of variables

changes the angle between asymptotes

and, hence, it can

diminish the winding number N

Example: polynomial potentials

are interrelated, $\alpha = 1/2$ giving $V_g(y)$ from

$$-\frac{d^2}{dx^2}\,\varphi(x)+\frac{\ell(\ell+1)}{x^2}\,\varphi(x)+V_f(x)\,\varphi(x) = E\,\varphi(x)\,,$$

$$V_f(x) = x^6 + f_4\,x^4 + f_2\,x^2 + f_{-2}\,x^{-2},$$

$$V_g(y) = -(iy)^2+i\,g_1\,y+g_{-1}\,(iy)^{-1}+g_{-2}\,(iy)^{-2}.$$

$\Longrightarrow$ upper sextic $\equiv$ rectified QT HO (pto)

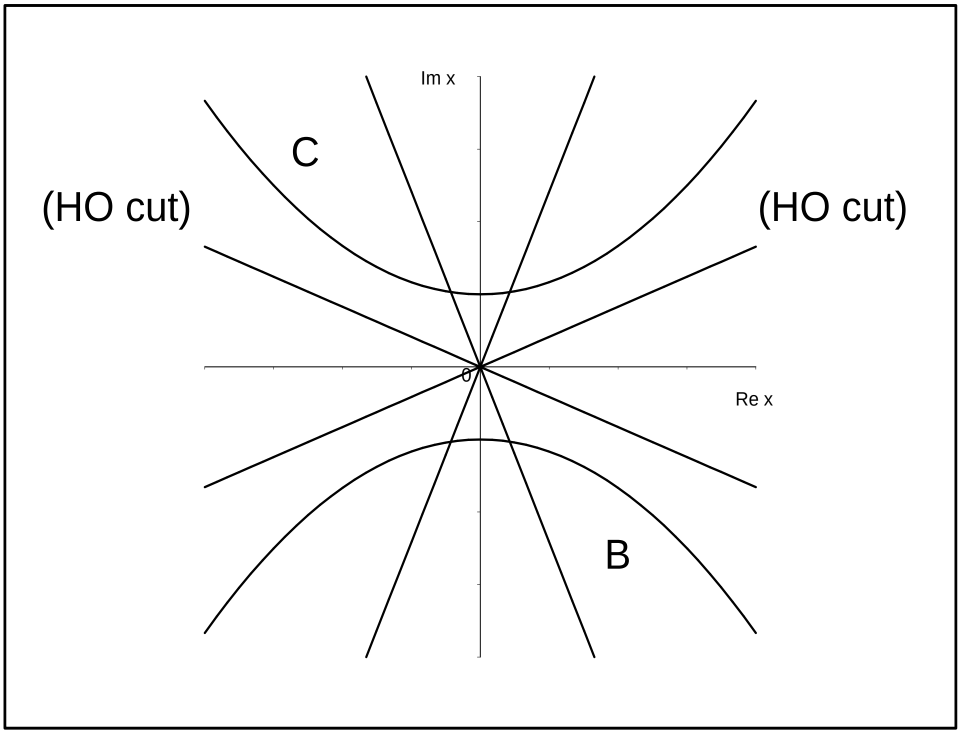

Figure 4: Sextic oscillators B (usual) and C (mapped on HO toboggan)

(c) feasible and useful:

perturbed harmonic oscillators living on a complex curve:

MZ (quant-ph/0606166v1):

Spiked harmonic quantum toboggans

Perturbed harmonic oscillator

$$V(x) = x^2 + \sum_\beta g_{(\beta)}\, x^\beta$$

can be **topologically nontrivial**. Its

$$\psi(x) \approx \psi^{(\pm)}(x) = e^{\pm x^2/2}\,, \qquad |x| \gg 1$$

= multivalued analytic functions

At any $k \in \mathbb{Z}$ they are

(a) "physical" (along a ray $x_\theta = \varrho e^{i\,\theta}$)

(b) "unphysical". E.g.,

$$\psi^{(-)}(x) = \begin{cases} \psi^{(phys)}(x), & k\pi + \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right), \\ \psi^{(unphys)}(x), & k\pi + \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right). \end{cases}$$

alternatively,

$$\psi^{(+)}(x) = \begin{cases} \psi^{(unphys)}(x), & k\pi + \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right), \\ \psi^{(phys)}(x), & k\pi + \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right). \end{cases}$$

For toboggans we define

$k_f = 0$ and $k_i = 1$ at $N = 0$,

$k_f = -1$ and $k_i = 2$ at $N = 1$,

$k_f = -2$ and $k_i = 3$ at $N = 2$ etc.

Riemann-surface "tobogganic trajectories"

$$\mathcal{D}_{(\varepsilon,N)}^{(PTSQM,\,tobogganic)} = \left\{ x = \varepsilon\, \varrho(\varphi, N)\, e^{i\,\varphi} \right\}$$

$$\varphi \in (-(N+1)\pi,\ N\pi)$$

$$\varrho(\varphi, N) = \sqrt{1 + \tan^2 \frac{\varphi + \pi/2}{2N+1}}$$

$\mathcal{PT}-$symmetry in the presence of the single branch point

.

parity-like operators $\mathcal{P}^{(\pm)} : x \to x{\cdot}\exp(\pm i\pi)$

map $\mathcal{K}_n$ into sheets $\mathcal{K}_{n\pm1}$.

two eligible rotation-type innovations $\mathcal{T}^{(\pm)}$

same for $\mathcal{P}^{(\pm)}\mathcal{T}^{(\pm)}$ and

$$\left(\mathcal{C}^{(N)}\right)^{\dagger} = \mathcal{D}_{(\varepsilon',N)}^{(PTSQM,\,tobogganic)}\,, \qquad \varepsilon' = \varepsilon\cdot e^{\pm i\pi}\,.$$

Bound states

$$H_{(\mathcal{PT})}\,\psi(x) = E\,\psi(x)$$

with Dirichlet inside the wedges,

$$\psi\left(\varrho\cdot e^{i\,\theta}\right) = 0, \qquad \varrho \gg 1 \quad \theta{+}k_{i,f}\,\pi \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

spectra = real in unbroken cases.

IV. SCATTERING ALONG THE TOBOGGANS

once "in" and "out" wedge boundaries are

$$\mathcal{A}_{(L)}^{(N)} \to \varrho\, e^{i\,\theta_{in}}, \qquad \theta_{in} = -(N+3/4)\,\pi,$$

$$\mathcal{A}_{(L)}^{(N)} \to \varrho\, e^{i\,\theta_{out}}, \qquad \theta_{out} = (N-1/4)\,\pi,$$

$$\mathcal{A}_{(U)}^{(N)} \to \varrho\, e^{i\,\theta_{in}}, \qquad \theta_{in} = -(N+5/4)\,\pi,$$

$$\mathcal{A}_{(U)}^{(N)} \to \varrho\, e^{i\,\theta_{out}}, \qquad \theta_{out} = (N+1/4)\,\pi\,.$$

independent solutions become equally large

and oscillate

not only when $V(x) < E$ at $\varrho \to \infty$

but also for many other potentials

including our x^2-dominated sample model.

incoming-beam normalization

$$\psi\left(\varrho\cdot e^{i\,\theta_{in}}\right)=\psi_{(i)}(x)+B\,\psi_{(r)}(x),\quad \varrho\gg 1,$$

and outcoming-beam normalization,

$$\psi\left(\varrho\cdot e^{i\,\theta_{out}}\right)=(1+F)\,\psi_{(t)}(x),\qquad \varrho\gg 1,$$

with incident and reflected waves

$$\psi_{(i,r)}(x)\approx e^{\pm i\varrho^2/2}.$$

Exactly solvable model of scattering on x^2

$$\left[-\frac{d^2}{dx^2}+\frac{\alpha^2-1/4}{x^2}+x^2\right]\psi(x)=E\,\psi(x),$$

set $x^2=-ir$ along the first nontrivial scattering path $\mathcal{A}_{(L)}^{(0)}$.

“in” branch with $r \ll -1$ and

“out” branch with $r \gg +1$

$$\chi_{(\alpha)}(r) = r^{\frac{1}{4}+\frac{\alpha}{2}}\, e^{ir/2}\, {}_1F_1\left(\frac{\alpha+1-\mu}{2}, \alpha+1; -ir\right),$$

linearly independent partner

$\chi_{(-\alpha)}(r)$, $\alpha \neq n \in I\!N$, $\qquad E = 2\mu$.

$|r| \gg 1$ estimate,

$$r^{\frac{1}{4}+\frac{\alpha}{2}}\chi_{(\alpha)}(r) \approx \; e^{ir/2}\,\frac{r^{\mu/2}\;\exp\left[-i\,\pi\;(\alpha+1)\,/4\right]}{\Gamma\left[(\alpha+1+\mu)/2\right]}+$$

$$+e^{-ir/2}\,\frac{r^{-\mu/2}\;\exp\left[+i\,\pi\;(\alpha+1)\,/4\right]}{\Gamma\left[(\alpha+1-\mu)/2\right]}\,.$$

"rigid" at $\alpha > 0$, $\mu = E/2 > 0$ and

$|x| = |\sqrt(r)| \gg 1$

Note that $\psi_{out}^{(Coul)}(r)$ becomes "distorted",

$$\sin(\kappa r{+}const) \to \sin(\kappa r{+}const{\cdot}\log r{+}const)\,.$$

Similar here, for $\psi_{in,out}(x) \approx$

$$r^{-1/4+(\alpha+\mu)/2}\, e^{ir/2}\, \frac{\exp\left[-i\,\pi\ \left(-\alpha+1\right)/4\right]}{\Gamma\left[(-\alpha+1+\mu)/2\right]} + \ldots\,.$$

V. TOBOGGANS IN POTENTIALS WITH MORE SPIKES

choose **two branch points** $x = \pm 1$,

$$V(x) = V_{regular}(x) + \frac{G}{(x-1)^2} + \frac{G^*}{(x+1)^2}$$

(cf. Sinha A and Roy P 2004 Czechosl. J. Phys. 54 129)

$\Longrightarrow$ **plus:** particle moving along a $\mathcal{PT}-$symmetric "toboggan" path.

(a) an enumeration of the paths

$x^{(QT)}(s)$ **encircling two branch points by winding**

- counterclockwise around $x_{(-)}^{(BP)}$ (letter L),
- counterclockwise around $x_{(+)}^{(BP)}$ (letter R),
- clockwise around $x_{(-)}^{(BP)}$ ($Q = L^{-1}$),
- clockwise around $x_{(+)}^{(BP)}$ ($P = R^{-1}$).

four-letter alphabet,

$x = x^{(\varrho)}(s)$,

a word ϱ of length $2N$,

$\varrho = \emptyset =$ non-tobogganic

$\mathcal{PT}-$symmetry $L \leftrightarrow R$, $\varrho = \Omega \cup \Omega^T$

$\boxed{\text{at } N=1}$, $\exists$ four possibilities,

$$\Omega \in \left\{L\,,L^{-1}\,,R\,,R^{-1}\right\}\,, \qquad N=1\,,$$

$$\varrho \in \left\{LR\,,L^{-1}R^{-1}\,,RL\,,R^{-1}L^{-1}\right\}\,, \quad N=1\,.$$

dozen cases $\boxed{\text{at } N = 2,}$

$$\Big\{LL, LR, RL, RR, L^{-1}R, R^{-1}L, LR^{-1},$$

$$R\,L^{-1}, L^{-1}L^{-1}, L^{-1}R^{-1}, R^{-1}L^{-1}, R^{-1}R^{-1}\Big\}$$

("shorter" $LL^{-1}, L^{-1}L, RR^{-1}, R^{-1}R$

not allowed among $4^2 = 16$ eligible)

$\boxed{\text{at } N = 3,}$ total number $= 36$:

cross 28 out of $4^3 = 64$ words,

$\Omega^{(NA)} = \Omega^{(NAL)} \cup \Omega^{(NAR)}$ (prev. L, R)

$\Omega^{(NAL)} = \Omega^{(NAL3)} \cup \Omega^{(NAL2)}$

one or two inversions in $\Omega^{(NAL3)}$ (six words),

in $\Omega^{(NAL2)}$ add R or R^{-1} (eight words).

at $N = 4$ we have $256 - 76 - 40 = 140$:

14 elements in $\Omega^{(NAL4)}$

24 elements in $\Omega^{(NAL3)}$, $L \leftrightarrow R$

$\Omega^{(NAL21)}$ (single inversion, 16 elements),

$\Omega^{(NAL22)}$ (two inversions, 8 elements)

$\Omega^{(NAL23)}$ (three inversions, 16 elements).

(b) rectifiable contours $x^{(\varrho_0)}(s)$

recollect: $\mathrm{i\,x} = (\mathrm{i\,z})^2\,,\ \psi_\mathrm{n}(\mathrm{x}) = \sqrt{\mathrm{z}}\,\varphi_\mathrm{n}(\mathrm{z})$

a *strict equivalence* of HO QT to

$$\left(-\frac{d^2}{dz^2} + 4z^6 + 4E_n z^2 + \frac{4\alpha^2 - 1/4}{z^2}\right)\varphi(z) = 0$$

along a *manifestly non-tobogganic* path.

•

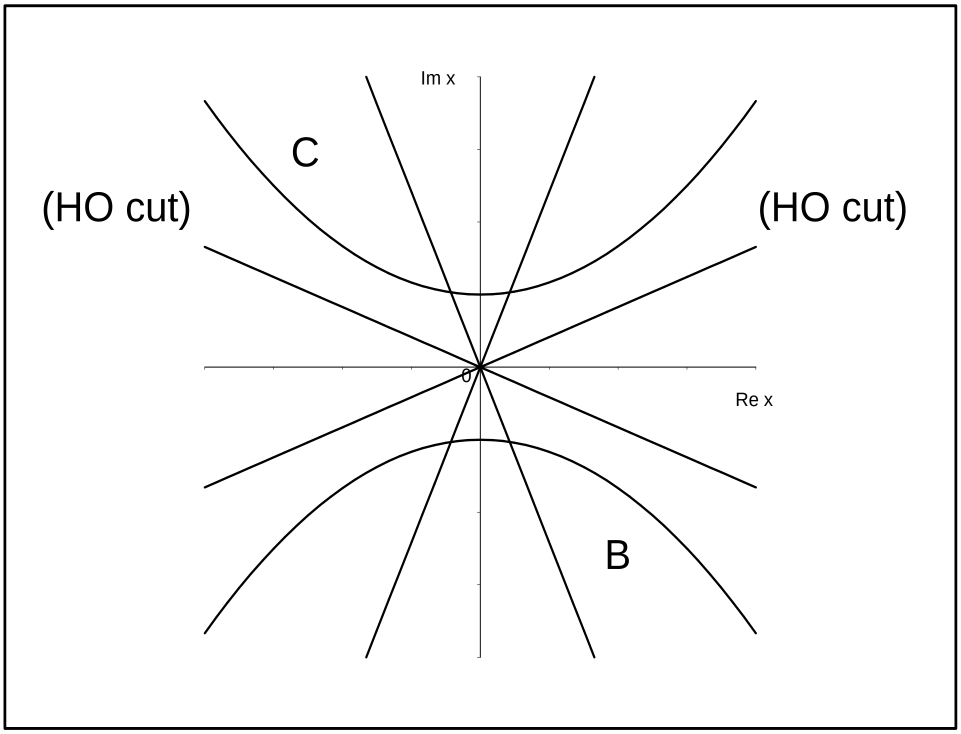

Figure 5: Both the HO-cut lines move upwards, contour C becomes tobogganic

now, to

$$\left[-\frac{d^2}{dx^2}+\frac{\ell(\ell+1)}{(x-1)^2}+\frac{\ell(\ell+1)}{(x+1)^2}+V(ix)\right]\psi(x)=$$

$$=E\,\psi(x)\,.$$

we, similarly, assign the rectified partner

$$\left[-\frac{d^2}{dz^2}+U_{eff}(\mathrm{i\,z})\right]\varphi(z)=0$$

$$U_{eff}(\mathrm{i\,z}) = \mathrm{U}(\mathrm{i\,z}) + \frac{\mu(\mu+1)}{(\mathrm{z}-1)^2} + \frac{\mu(\mu+1)}{(\mathrm{z}+1)^2} \;\equiv$$

$$\equiv\; U(\mathrm{i\,z}) + 2\,\frac{\mu(\mu+1)[1-(\mathrm{i\,z})^2]}{\left[1+(\mathrm{i\,z})^2\right]^2}\,.$$

implicit rectification formula

$$1 + (ix)^2 = \left[1 + (iz)^2\right]^\kappa \,, \qquad \kappa > 1$$

$z = -\mathrm{i}\,\varrho$ on itself:

explicit rectification formula

$$x = -\mathrm{i}\sqrt{(1 - \mathrm{z}^2)^\kappa - 1}$$

Effective non-tobogganic potentials - construction = routine:

$$\frac{d}{d\,x} = \beta(z)\,\frac{d}{d\,z}\,, \qquad \beta(z) = -\mathrm{i}\,\frac{\sqrt{(1-\mathrm{z}^2)^\kappa - 1}}{\kappa\,\mathrm{z}\,(1-\mathrm{z}^2)^{\kappa-1}}\,.$$

$\psi(x) = \chi(z)\,\varphi(z)$ with $\chi(z) = const\,/\sqrt{\beta(z)}$

(Liouville L 1837 J.Math.Pures Appl. 1 16)

$$\left(-\beta(z)\,\frac{d}{dz}\,\beta(z)\,\frac{d}{dz} + V_{eff}[ix(z)] - E\right)\,\chi(z)\,\varphi(z) = 0$$

$$U_{eff}(\mathrm{i}\,\mathrm{z}) = \frac{\mathrm{V_{eff}[i\,x(z)] - E_n}}{\beta^2(\mathrm{z})} + \frac{\beta''(\mathrm{z})}{2\,\beta(\mathrm{z})} - \frac{[\beta'(\mathrm{z})]^2}{4\,\beta^2(\mathrm{z})}$$

QED

shapes of the tobogganic pull-backs:

the vicinity of the negative imaginary axis

$$z=-\mathrm{i}\,\mathrm{r}\,\mathrm{e}^{\mathrm{i}\,\theta}\ \longrightarrow\ \mathrm{x}=-\mathrm{i}\,\left[\left(1+\mathrm{r}^2\,\mathrm{e}^{2\,\mathrm{i}\,\theta}\right)^{\kappa}-1\right]^{1/2}\,.$$

factor $\sqrt{\kappa}$ at the small radii r

parallelism at $r \gg 1$

consequences:

knot-like $x^{\varrho_0}(s)$ by computer graphics,

straight-line $z(s) = s - \mathrm{i}\,\varepsilon$ pulled back

N sensitive to ε (pto)

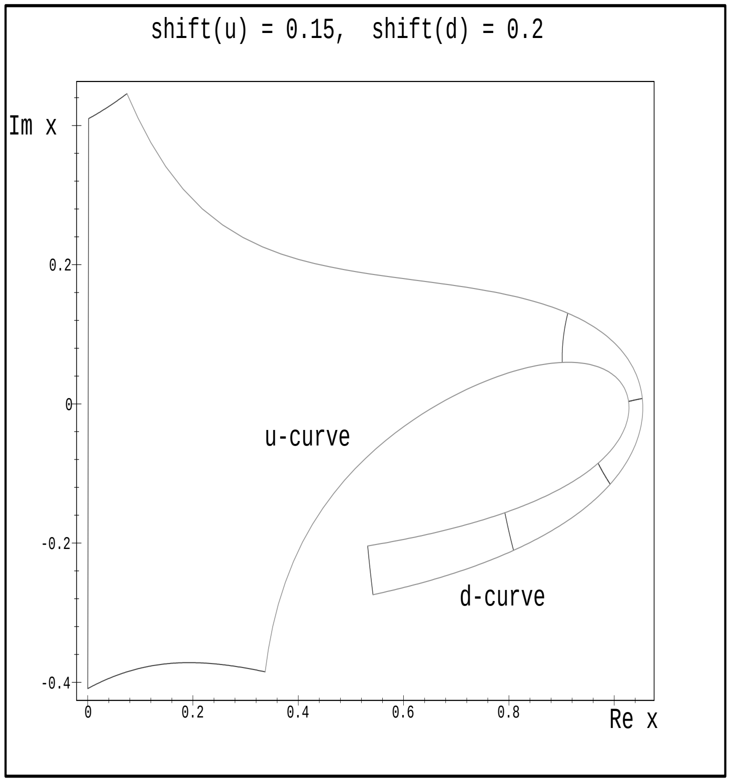

Figure 6: Two bitoboggans ($\kappa = 2.4$, $s \in (0.4, 1.4)$)

favorable property:

winding number grows *quickly* with κ

(pto)

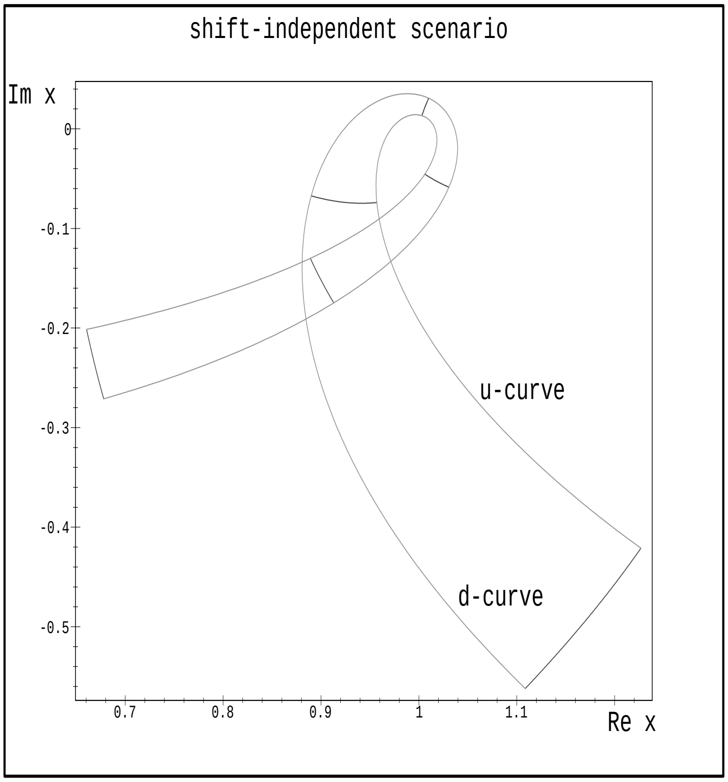

Figure 7: Two bitoboggans ($\kappa = 3$, $s \in (0.4, 1.4)$)

user-friendliness:

winding numbers *arbitrarily* large

paths *very close* to BPs

the sensitivity to the shift *recurs*

(pto)

•

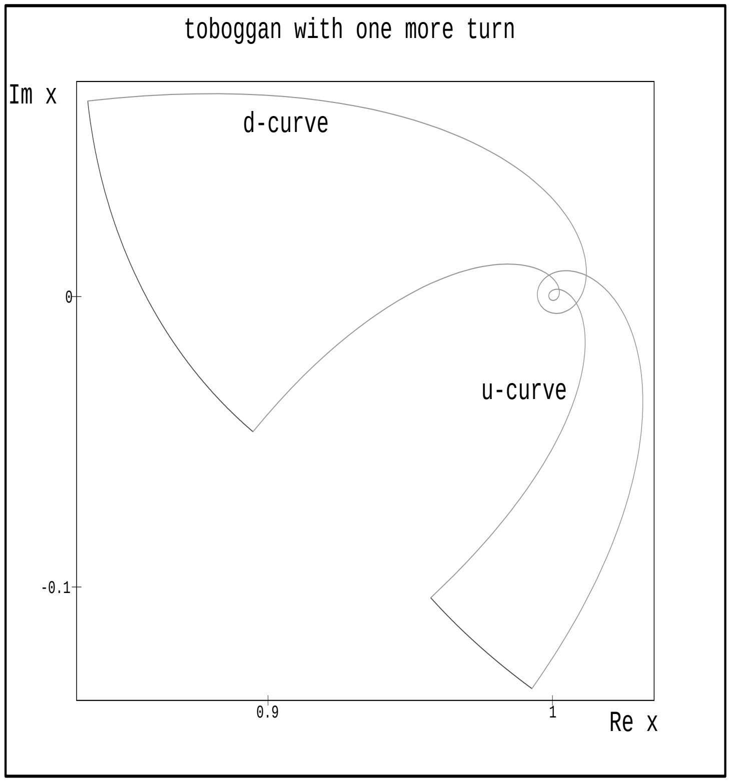

Figure 8: Two bitoboggans ($\kappa = 5$, $s \in (0.4, 1.4)$)

VI. SUMMARY

- in rectified tobogganic contours $x^{(\varrho_0)}(s)$

  descriptor word ϱ_0 inferred *a posteriori*

- QT2 observable *if and only if*

  $\mathcal{PT}-$symmetry unbroken

- topology-dependent spectra