# NON-HERMITIAN REPRESENTATIONS OF OBSERVABLES: A REVIEW OF RECENT PROGRESS

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**Superintegrable Systems in Classical and Quantum Mechanics,**

**FNSPE CTU, Prague, Tuesday, May 6th, 2008, 12.00 pm**

**brief outline of $\mathcal{PT}-$symmetric Quantum Mechanics:**

- **I.** *prelude:* context in **physics**

- **II. exposition**, puzzle: $\exists$ *non-Hermitian* Hamiltonians (cf. $H = p^2 + \mathrm{i}x^3$) with *real* ($\Rightarrow$ observable) spectra!

- **III. explanation***:* "Fourier" (unitary) vs. "Dyson" ($\Rightarrow$ non-unitary *representations* made *easy* )

- **IV. new applications** of the *cryptohermiticity* (cf. $H = H(t)$, $\Rightarrow$ use easy kets and difficult bras!)

# Part I. BROADER CONTEXT IN PHYSICS

(what I am NOT going to explain today)

(just further reading recommended briefly)

**A. we shall consider just one observable: Hamiltonian** $H$

- add spin: M. Z., J. Phys. A: Math. Gen. 39 (2006) 441

- add asymptotic coordinate: M. Z., submitted

**B. we shall stay non-relativistic:** $H = -\triangle + V$

- move to Klein Gordon: M. Z., H. Bíla, V. Jakubský,

Czech. J. Phys. 54 (2004) 1143.

- move to Proca: J. Smejkal, V. Jakubský, M. Z.,
J. Phys. Studies 11 (2007) 45.

**C. we shall consider just 1D space:**

• easy generalization for separable:

(M. Z., J. Phys. A: Math. Gen. 36 (2003) 7825)

**D. we shall consider just 1P degrees of freedom:**

• $\exists$ $A = 3$ Calogero with one parameter (M. Z. and M. Tater, J. Phys. A: Math. Gen. 34 (2001) 1793) or more parameters (A. Fring and M. Z., ibid. 41 (2008) 194010).

**E. we shall skip illustrations in nuclear physics:**

• Dyson's bosonic images of nuclei (IBM):

F. G. Scholtz, H. B. Geyer and F. J. W. Hahne,

Ann. Phys. 213 (1992) 74

**F. we shall skip illustrations in field theory:**

• ghost busting in Lee model, etc:

Carl M. Bender, Rep. Prog. Phys. 70 (2007) 947.

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# Part II. A FEW HO-TYPE EXAMPLES

(a): motivation: Quantum Mechanics' warm up

(b): aim: puzzle exposed, feasibility illustrated

## A. the first example: Bender and Boettcher

- *trivial* HO Schrödinger equation with *real* $E = 2n + 1$,

$$-\frac{d^2}{dx^2}\,\psi(x) + x^2\,\psi(x) = E\,\psi(x)$$

- Hermite-polynomials-solvable at all $x \in \mathbb{C}$

  **defined along the straight contour**

$$\mathcal{C}^{(BG)} = \{x \mid x = s - i\,\varepsilon,\; s \in \mathbb{R}\}$$

## B. the second example: spiked HO

$$\left(-\frac{d^2}{dx^2} + \frac{\ell(\ell+1)}{x^2} + x^2\right)\,\psi(x) = E\,\psi(x)$$

same contour, $\exists$ "twice as many" bound states,

$$E = E_{n,\ell,\pm} = 4n + 2 \pm (2\ell + 1) = \textbf{all real}$$

see M. Z., *PT symmetric harmonic oscillators*

Phys. Lett. A 259 (1999) 220 - 3.

## C. the third example: toboggans

Miloslav Znojil (quant-ph/0502041):

*PT-symmetric quantum toboggans*

Phys. Lett. A 342 (2005) 36-47.

**(topology in application**

**and monodromy group in application)**

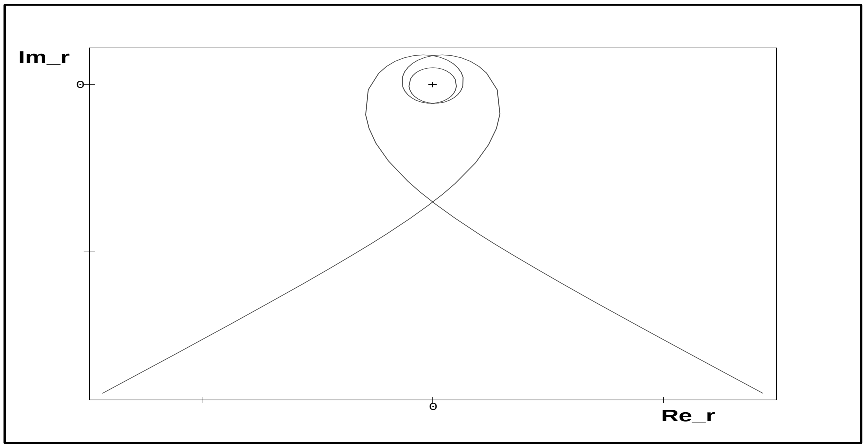


Figure 1: Complex straight-line contour $\mathcal{C}^{(BG)}$ generalized (tobogganic)

## D. the fourth example: quantum knots

• **free** radial Schrödinger equations with $n = 0, 1, \ldots$ in

$$-\frac{d^2}{dr^2}\,\psi(r) + \frac{\ell(\ell+1)}{r^2}\,\psi(r) = E\,\psi(r)\,, \qquad \ell = n + \frac{D-3}{2}$$

$$E = \kappa^2,\ z = \kappa r \text{ and } \psi(r) = \sqrt{z}\,\varphi(z)$$

• Bessel – solvable:

$$\psi(r) = c_1\,\sqrt{r}\,H_\nu^{(1)}(\kappa\, r) + c_2\,\sqrt{r}\,H_\nu^{(2)}(\kappa\, r)\,, \qquad \nu = \ell + 1/2\,.$$

**asymptotic wedges = defined via angles:**

on the multisheeted Riemann surface of multivalued analytic

wave functions $\psi(r)$

- $\mathcal{S}_0 = \{r = -\mathrm{i}\,\varrho\, e^{\mathrm{i}\,\varphi} \,|\, \varrho \gg 1\,,\ \varphi \in (-\pi/2, \pi/2)\}$,
- $\mathcal{S}_{\pm k} = \{r = -\mathrm{i}\, e^{\pm\mathrm{i}\,k\,\pi}\,\varrho\, e^{\mathrm{i}\,\varphi} \,|\, \varrho \gg 1\,,\ \varphi \in (-\pi/2, \pi/2)\}$
- solved, with $\mathcal{C}^{(N)}$ connecting $\mathcal{S}_0$ and $\mathcal{S}_m$, $m = 2N$

M. Z., *Quantum knots*, Phys. Lett. A 372 (2008) 3591-6:

$$H_\nu^{(2)}\left(ze^{\mathrm{i}m\pi}\right) = \frac{\sin(1+m)\pi\nu}{\sin\pi\nu}\, H_\nu^{(2)}(z) + e^{\mathrm{i}\pi\nu}\,\frac{\sin m\pi\nu}{\sin\pi\nu}\, H_\nu^{(1)}(z)$$

• **solved at any energy** $E = \kappa^2$, since

boundary conditions **quantize the angular momenta**:

$$2N\nu = integer\,, \qquad \nu \neq integer \implies \ell = \frac{M-N}{2N}\,,$$

$M = 1, 2, 3, \ldots$ , **with forbidden** $M \neq 2N, 4N, 6N, \ldots$ .

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## Part III. THEORY

(a): motivation: Hermiticity sacrificed $\Leftrightarrow$ simplicity gained

(b): aim: feasibility should be achieved in *new models*

## $\odot$ A NAIVE SUMMARY:

- a redefinition of the inner product in the Hilbert space is

needed – replace

$$\langle \psi \,|\, \phi \rangle = \int \psi^*(x)\phi(x)dx$$

by

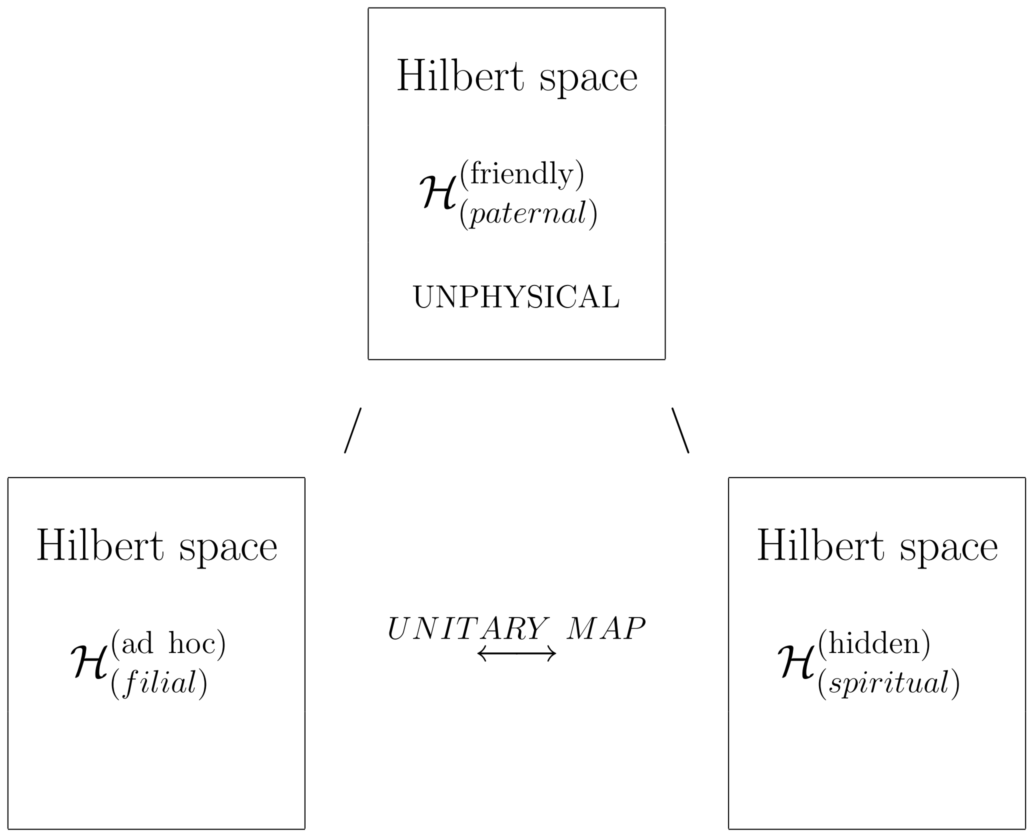
$$\langle \psi \,|\, \phi \rangle = \int \psi^*(x)\,\Theta(x, y)\,\phi(y)dx\,dy\,.$$

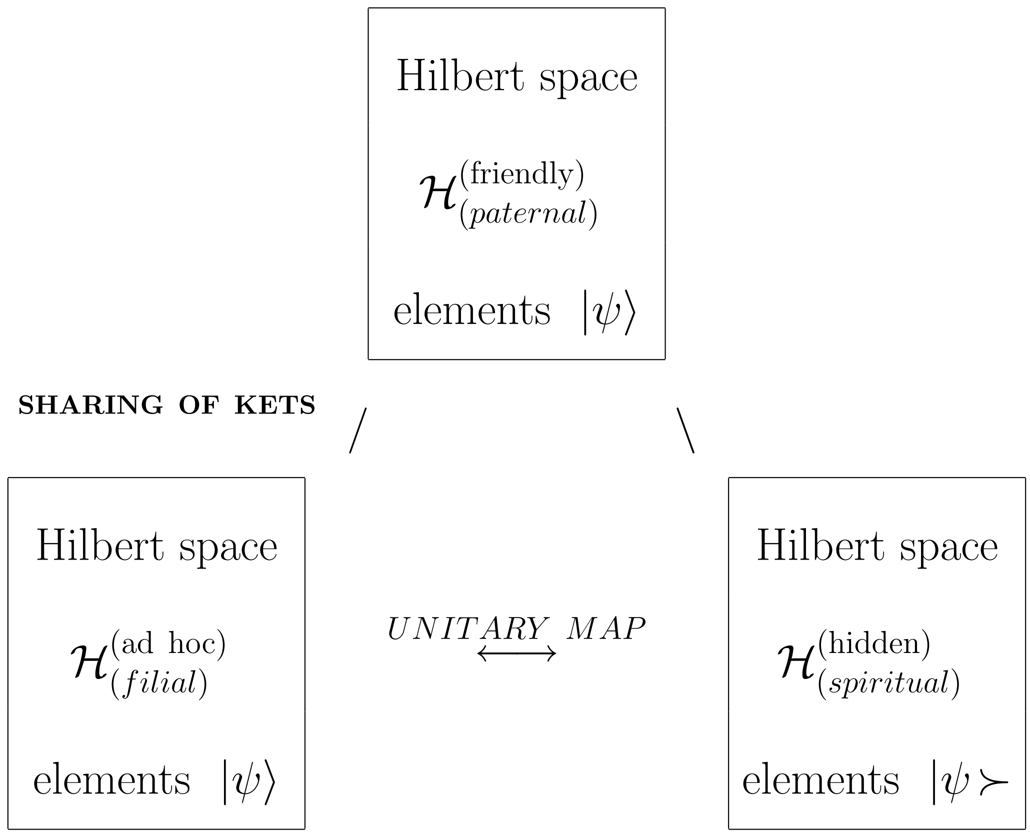
## $\otimes$ A LESS NAIVE APPROACH:

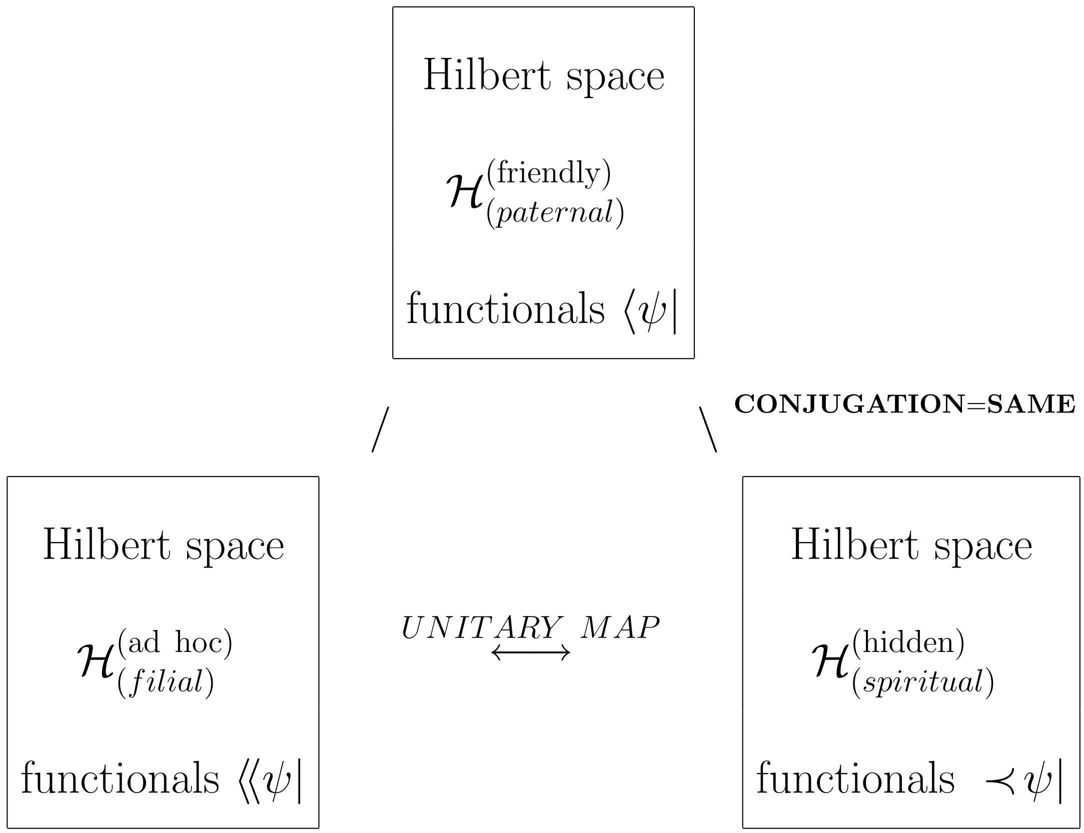
*three-Hilbert-space formulation of QM*

M.Z., Phys. Lett. A 372 (2008) 3591-6

main idea: P.T.O.


Hilbert space
$\mathcal{H}^{(\text{friendly})}_{(paternal)}$
UNPHYSICAL
Hilbert space
$\mathcal{H}^{(\text{ad hoc})}_{(filial)}$
UNITARY MAP
Hilbert space
$\mathcal{H}^{(\text{hidden})}_{(spiritual)}$


Hilbert space
$\mathcal{H}^{(\text{friendly})}_{(paternal)}$
elements $|\psi\rangle$
SHARING OF KETS
Hilbert space
$\mathcal{H}^{(\text{ad hoc})}_{(filial)}$
elements $|\psi\rangle$
UNITARY MAP
Hilbert space
$\mathcal{H}^{(\text{hidden})}_{(spiritual)}$
elements $|\psi \succ$


Hilbert space
$\mathcal{H}^{(\text{friendly})}_{(paternal)}$
functionals $\langle\psi|$
CONJUGATION=SAME
Hilbert space
$\mathcal{H}^{(\text{ad hoc})}_{(filial)}$
functionals $\langle\!\langle\psi|$
$\overset{UNITARY\ MAP}{\longleftrightarrow}$
Hilbert space
$\mathcal{H}^{(\text{hidden})}_{(spiritual)}$
functionals $\prec\psi|$

## the key definition

$$\langle\langle\psi| = \langle\psi|\,\Theta$$

F. G. Scholtz, H. B. Geyer and F. J. W. Hahne,

Ann. Phys. (NY) 213 (1992) 74

**meaning:** nonstandard Hermitian conjugation

confusing: practically NEVER used

## the other definitions

Table 1: The triplet of Hilbert spaces in quantum mechanics

| space | element | **dual** | *inner product* | Hamiltonian |
|---|---|---|---|---|
| paternal | $\vert\psi\rangle$ | $\langle\psi\vert = (\vert\psi\rangle)^\dagger$ | $\langle\psi\vert\psi'\rangle$ | $(H)^\dagger \neq H$ |
| filial | $\vert\psi\rangle$ | $\langle\langle\psi\vert \equiv \prec\psi\vert\Omega$ | $\langle\langle\psi\vert\psi'\rangle$ | $\left[H = (H)^\ddagger\right]$ |
| spiritual | $\vert\psi\succ \equiv \Omega\vert\psi\rangle$ | $\prec\psi\vert = \langle\psi\vert\Omega^\dagger$ | $\prec\psi\vert\psi'\succ$ | $h = (h)^\dagger$ |

invertible map $\Omega \neq \Omega^\dagger$, positive metric $\Theta = \Omega^\dagger\Omega$

operators of observables $h = \Omega\ H\ \Omega^{-1}$, $H^\dagger = \Theta\, H\, \Theta^{-1}$

## $\odot$ SUMMARY OF THE THEORY:

- we **must** work in physical $\mathcal{H}^{(filial)}$ with product

$$\langle\!\langle \psi \,|\, \phi \rangle = \int \psi^*(x)\,\Theta(x,y)\,\phi(y) dx\, dy$$

- in the Hilbert space $\mathcal{H}^{(paternal)}$ the popular "redefinition"

philosophy can prove misleading when $\Omega = \Omega(t)$:

cf. M.Z., arXiv 0711.0535

"Which operator generates time evolution in Q. Mechanics?"