.

Coupling of channels in PT-symmetric models

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I. A coupling of channels?

It's always been there!

Just recollect:

(a) Bender's PT symmetric potentials

$$V(x) = V_{symm}(x) + \mathrm{i}\, V_{antisymm}(x),$$

$$V_{symm}^{(example)}(x) = \omega^2 x^2 + \lambda^2 x^4,$$

$$V_{antisymm}^{(example)}(x) = g\, x^3$$

in HO basis $|n, \pm\rangle$

$$H = \begin{pmatrix} S & B \\ -B^T & L \end{pmatrix}, \qquad P = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

channels, decoupled iff $g \to 0$.

(b) relativistic Sakata - Taketani:

$$H = \begin{pmatrix} 0 & K \\ \\ I & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & I \\ \\ I & 0 \end{pmatrix}$$

in fixed-frame evolution:

$$H \begin{pmatrix} x \\ \\ y \end{pmatrix} = E \begin{pmatrix} x \\ \\ y \end{pmatrix}$$

put $x = E\,y$ and reduce to $K\,y = E^2\,y$

require $E^2 > 0$, **define** $E_\pm = \pm\sqrt{E^2}$

and rotate to the **FV channels**.

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II. Toy model with two coupled channels

(a) Hamiltonian:

$$H_{(kinetic)} = \begin{pmatrix} -\frac{d^2}{dx^2} & 0 \\ & \\ 0 & -\frac{d^2}{dx^2} \end{pmatrix},$$

$$H_{(interaction)} = \begin{pmatrix} V_a(x) & W_b(x) \\ & \\ W_a(x) & V_b(x) \end{pmatrix}.$$

(b) its $\theta-$pseudo-Hermiticity:

$$\theta = \theta^{\dagger} = \begin{pmatrix} 0 & \mathcal{P} \\ & \\ \mathcal{P} & 0 \end{pmatrix} = \theta^{-1}$$

(c) potentials $[x \in (-1, 0)]$:

$$\mathrm{Im}\, W_a(x) = X > 0,$$

$$\mathrm{Im}\, W_b(x) = Y > 0,$$

$$\mathrm{Im}\, V_a(x) = \mathrm{Im}\, V_b(x) = Z,$$

(d) spin-like ($\sigma = \pm 1$) symmetry:

$$\Omega = \left(\begin{array}{cc} 0 & \omega^{-1} \\ \omega & 0 \end{array} \right), \qquad \omega = \sqrt{\frac{X}{Y}} > 0.$$

(e) solvable (details see below)

(f) physical (details see below)

(g) simple in a modified Dirac's notation

$$H|E,\sigma\rangle = E|E,\sigma\rangle, \quad \Omega\,|E,\sigma\rangle = \sigma\,|E,\sigma\rangle$$

$$\langle\!\langle E,\sigma|H = E\langle\!\langle E,\sigma|, \quad \langle\!\langle E,\sigma|\Omega = \sigma\langle\!\langle E,\sigma|$$

$$\text{biog.}: \quad 0 = \langle\!\langle E',\sigma'|E,\sigma\rangle \times \begin{cases} (E'-E) \\ \\ (\sigma'-\sigma) \end{cases}$$

$$\text{cpl.}: I = \sum_{E,\sigma} |E,\sigma\rangle \frac{1}{\langle\!\langle E,\sigma|E,\sigma\rangle} \langle\!\langle E,\sigma|$$

$$\text{sp.}: \quad H = \sum_{E,\sigma} |E,\sigma\rangle \frac{E}{\langle\!\langle E,\sigma|E,\sigma\rangle}\langle\!\langle E,\sigma|$$

$$\Omega = \sum_{E,\sigma} |E,\sigma\rangle \frac{\sigma}{\langle\!\langle E,\sigma|E,\sigma\rangle}\langle\!\langle E,\sigma|$$

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III. Key problem:

Shall we be able to introduce the SGH metric?

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Subsummary:

Introducing a “physical”’ metric?

Perhaps only too easily!

The sleepers during the next two screens

will be also given the message

by the next speaker.

(a) cc metric Θ for students: 2×2:

$$H \to \begin{pmatrix} -T & B \\ & \\ -B & T \end{pmatrix}, \qquad \Theta = \begin{pmatrix} a & b \\ & \\ b & d \end{pmatrix}$$

$$\Theta H = H^T \Theta \Longrightarrow 2bT = -B(a+d)$$

$E \in I\!R \Longleftrightarrow |T| \geq |B|$, $B = T\, \sin \alpha$

$\theta_{1,2} > 0 \Longleftrightarrow b \neq 0 \neq a + d = 2Z$

and for $a = Z(1+\xi)$, $d = Z(1-\xi)$,

$$1 > \sqrt{\xi^2 + \sin^2 \alpha}.$$

We have an **interval** of $\xi < \cos \alpha$ (!)

(b) biorthogonal “brabraket” basis:

$$\langle\langle n|\, H = \langle\langle n|\, E_n, \qquad H\,|n\rangle = E_n\,|n\rangle$$

+ the Mostafazadeh's universal formula:

$$\Theta = \mathbf{\Sigma}\ |n\rangle\rangle\, s_n\, \langle\langle n|\,, \qquad s_k > 0\,.$$

KG **cc = direct sum** of 2×2 matrices:

$$H = \begin{pmatrix} 0 & B \\ & \\ I & 0 \end{pmatrix}, \qquad \Theta = \begin{pmatrix} a & b \\ & \\ b & d \end{pmatrix},$$

gives $\theta_{1,2} > 0$ for all $d = aE^2 > 0$,

i.e., **full intervals** of $|b_n| < |a_n E_n|$

(recommended choice: $a = 1/E$).

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IV. Another subsummary:

Square wells with $x \in (-1, 1)$?

$$V(x) = V_{(Z)}(x) = -\mathrm{i}\, Z \,\mathrm{sign}(x)?$$

Why at all?

(a) = **ODE** with constant coefficients:

$$-\frac{d^2}{dx^2}\,\varphi^{(m)}(x) + \sum_{j=1}^{K} V_{Z_{(m,j)}}(x)\,\varphi^{(j)}(x) =$$

$$= E\varphi^{(m)}(x), \qquad m = 1, 2, \ldots, K$$

= **solvable** by an ansatz for $\varphi^{(m)}(x)$

$$= \begin{cases} C_L^{(m)} \sin \kappa_L(x+1), & x < 0, \\ C_R^{(m)} \sin \kappa_R(-x+1), & x > 0 \end{cases}$$

= **giving** $Z_{(eff)}^{(m)}(K)$ as eigenvalues of

$$\begin{pmatrix} Z_{(1,1)} & Z_{(1,2)} & \cdots & Z_{(1,K)} \\ Z_{(2,1)} & Z_{(2,2)} & \cdots & Z_{(2,K)} \\ \vdots & \ddots & \ddots & \vdots \\ Z_{(K,1)} & Z_{(K,2)} & \cdots & Z_{(K,K)} \end{pmatrix}.$$

(b) quantized easily:

$= \text{ansatz} \rightarrow \kappa_R = s + \mathrm{i}t = \kappa_L^*, \quad s > 0,$

$\rightarrow t = t_{first\ curve}(s) = Z_{(eff)}^{(m)}(K)/(2s)$

plus **matching** in the origin:

$\rightarrow \kappa_L \,\text{cotan}\, \kappa_L = -\kappa_R \,\text{cotan}\, \kappa_R$

gives the second, "universal" curve

$t = t_{exact}(s)$ with implicit definition

$$\boxed{2s\ \sin 2s + 2t\ \sinh 2t = 0}$$

$\rightarrow$ **energies** via **intersections**,

$$E_n = s_n^2 - t_n^2, \qquad n = 0, 1, \ldots\ .$$

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V. Technicalities are over at last.

Now there come THE IDEAS!

(a) the first idea: **relax** $\mathbf{P} = \mathbf{P}^\dagger$

pattern: if $H^\dagger = \mathbf{R}\, H\, \mathbf{R}^{-1}$ and $\mathbf{R} \neq \mathbf{R}^\dagger$,

we have the symmetry,

$$H\,\mathcal{S} = \mathcal{S}\,H, \qquad \mathcal{S} = [\mathbf{R}^{-1}]^\dagger \mathbf{R}.$$

Let's choose $\mathbf{R}^{-1} = \mathbf{R}^\dagger$ with $\mathcal{S} = \mathbf{R}^2$ and

$$\mathbf{R} = \begin{pmatrix} 0 & \ldots & 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & \ldots & 0 & 0 \\ 0 & \mathcal{P} & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \mathcal{P} & 0 \end{pmatrix}$$

at any K.

(b) the second idea: **take more**:

$\mathbf{R} = \mathbf{R}_{(K,L)} = \mathcal{P}\,\mathbf{r}_{(K,L)}$ such that

$$\mathbf{r}_{(K,L+1)} = \mathbf{r}_{(K,1)}\,\mathbf{r}_{(K,L)}\,, \quad L = 1, 2, \ldots .$$

$$\left[\mathbf{r}_{(K,L)}\right]^K = I,\ \mathbf{r}_{(K,K-L)} = \left[\mathbf{r}_{(K,L)}\right]^\dagger.$$

(c) necessity: **adapt** H to $\mathbf{R}$:

$$\mathbf{A} = \mathbf{r}_{(K,K-L)} \cdot \mathbf{A}^T \cdot \mathbf{r}_{(K,L)}$$

= solve a finite set of equations

(see the $K = 2$ result above)

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VI. Let us move now to

THREE channels, with

$$\mathbf{R}_{(3,1)} = \begin{pmatrix} 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & 0 \\ 0 & \mathcal{P} & 0 \end{pmatrix} = \mathbf{R}^{\dagger}_{(3,2)} = \mathbf{R}^{-1}_{(3,2)},$$

$$\mathbf{R}_{(3,2)} = \begin{pmatrix} 0 & \mathcal{P} & 0 \\ 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & 0 \end{pmatrix} = \mathbf{R}^{\dagger}_{(3,1)} = \mathbf{R}^{-1}_{(3,1)}.$$

giving the **unique**

$$\mathbf{A}_{(interaction)} = \begin{pmatrix} Z & X & X \\ X & Z & X \\ X & X & Z \end{pmatrix}, \quad L = 1, 2$$

and **solutions** with the 'first curve'

$$t = t^{(\sigma)}(s) = \frac{1}{2s} Z_{eff}(\sigma), \quad \sigma = 1, 2, 3$$

$$Z_{eff}(1) = Z + 2\,X \text{ and } Z_{eff}(2, 3) = Z - X$$

and with the **cc** coefficients

$$\left(C^{(a)}_{(1)}, C^{(b)}_{(1)}, C^{(c)}_{(1)}\right) \sim (1, 1, 1)$$

$$\left(C^{(a)}_{(2)}, C^{(b)}_{(2)}, C^{(c)}_{(2)}\right) \sim (1, -1, 0)$$

$$\left(C^{(a)}_{(3)}, C^{(b)}_{(3)}, C^{(c)}_{(3)}\right) \sim (1, 1, -2)\,.$$

The energies stay real in the 2D domain

$$Y - Z_{crit} \leq Z \leq Z_{crit} - 2Y.$$

[vertices $(0, \pm 4.475)$ and $(2.98, -1.49)$].

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VII. A bit of a technical interlude:

Numerics relevant!

Return, quickly, to $K = 1$!!

(a) weakly non-Hermitian regime:

$$s = s_n = \frac{(n+1)\pi}{2} + \tau \frac{Q_n}{2}, \quad \tau = (-1)^n$$

$\longrightarrow$ solvable by **iterations**:

the first small quantity $\varrho \equiv \frac{1}{L} = \frac{1}{(n+1)\pi}$

the second one $\alpha = \frac{2\, Z_{eff}(\sigma)}{L}$ or $\beta = \alpha \varrho$

$\longrightarrow$ a "generalized continued fraction"

$$Q = \arcsin\left(2t \frac{\varrho}{1+\tau\, Q\, \varrho} \sinh 2t\right),$$

$$\text{where} \quad 2t = \frac{\alpha}{1+\tau\, Q\, \varrho}$$

(b) intermediate non-Hermiticicities:

$\rightarrow$ *ad hoc* **perturbation theory**:

$\rightarrow \arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \ldots$

$Q = Q(\alpha,\beta) = \alpha\beta\,\Omega(\alpha,\beta),$

$\rightarrow \Omega(\alpha,\beta) = 1 + c_{10}\,\alpha^2 + c_{01}\,\beta^2 +$

$+ c_{20}\,\alpha^4 + c_{11}\,\alpha^2\beta^2 + c_{02}\,\beta^4 + \mathcal{O}(\alpha^6)$

$\rightarrow$ equation **re-arranged**:

$$[1 + \tau\,\beta^2\Omega(\alpha,\beta)]\,\mathrm{arcsinh}(\Lambda) = \alpha$$

$\Lambda = [1 + \tau\,\beta^2\Omega(\alpha,\beta)]^2\,\frac{1}{\beta}\,\sin[\alpha\beta\,\Omega(\alpha,\beta)]$

(c) formulae:

$\longrightarrow$ leading order relation

$$0 = \left(-\tfrac{1}{6} + c_{10} + c_{01}\,\varrho^2 + 3\tau\varrho^2\right)\alpha^3 + \ldots .$$

determines the first two coefficients,

$$c_{10} = \tfrac{1}{6}, \qquad c_{01} = -3\tau\,,$$

the next-order $O\left(\alpha^5\right)$ gives

$$c_{20} = \tfrac{1}{120}, \qquad c_{11} = \tfrac{1-8\tau}{6}, \qquad c_{02} = 15$$

and the $1 + O\left(\alpha^4\right)$ formula

$$Q_n = \frac{4\,Z_{eff}^2}{(n+1)^3\pi^3} + \\ + \frac{8\,Z_{eff}^4}{3\,(n+1)^5\pi^5}\left(1 + \frac{18\,(-1)^{n+1}}{(n+1)^2\pi^2}\right).$$

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VIII.

FOUR channels

$K = 4$ warning: $\mathbf{R}_{(4,2)}$ is Hermitian,

mere six constraints upon 16 couplings.

Not enough symmetry for us.

Unique coupling-matrix left,

$$\mathbf{A}_{(interaction)} = \begin{pmatrix} Z & U & D & U \\ L & Z & L & D \\ D & U & Z & U \\ L & D & L & Z \end{pmatrix}, \quad L = 1, 3.$$

solution :

Four shifts of the effective Z,

$[-D, -D, D + 2\sqrt{UL}, D - 2\sqrt{UL}]$

with respective eigenvectors

$$\{1, 0, -1, 0\}, \{0, 1, 0, -1\},$$

$$\left\{U, \pm\sqrt{UL}, U, \pm\sqrt{UL}\right\}.$$

remark:

from the pseudo-parity

$$\mathbf{r}^{(permuted)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

repartitioned model

$$\mathbf{A}^{(permuted)}_{(interaction)} = \left(\begin{array}{cc|cc} Z & D & U & U \\ D & Z & U & U \\ \hline L & L & Z & D \\ L & L & D & Z \end{array}\right), \quad L = 1, 3.$$

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IX.

FIVE channels

$$\mathbf{r}_{(5,1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \ldots,$$

all lead to the same

$$\mathbf{A}_{(interaction)} = \begin{pmatrix} Z & X & D & D & X \\ X & Z & X & D & D \\ D & X & Z & X & D \\ D & D & X & Z & X \\ X & D & D & X & Z \end{pmatrix}.$$

$\longrightarrow$ exceptional eigenvalue $F_0 = 2\,D + 2\,X$

giving eigenvector $\{1, 1, 1, 1, 1\}$

$\longrightarrow$ the reduced $Z = 0$ matrix A has the pair of the twice degenerate eigenvalues,

$$F_{\pm} = \frac{1}{2}\left[-D - X \pm \sqrt{5}\,(-D + X)\right]$$

with the two respective eigenvectors

$$\left\{1 \mp \sqrt{5}, -1 \pm \sqrt{5}, 2, 0, -2\right\}$$

$$\left\{1 \mp \sqrt{5}, -2, 0, 2, -1 \pm \sqrt{5}\right\}.$$

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X. Another important interlude concerning the domain where the energies remain real.

(a) a numerical **algorithm**:

$$\left.\frac{Q}{2}\right|_{crit} \equiv \varepsilon(t_{crit}) = \pi - \frac{Z_{crit}}{2t_{crit}},$$

$$\sin\left[2\,\varepsilon(t)\right] = \frac{t\ \sinh\ 2t}{\pi - \varepsilon(t)},$$

$\varepsilon_{(lower)}(t) = \pi/4$ and $\varepsilon_{(upper)}(t) = 0$.

$$\partial_t \varepsilon(t_{crit}) = \frac{Z_{crit}}{2t^2_{crit}},$$

$$\partial_t \varepsilon(t) = \frac{\sinh\ 2t + 2t\ \cosh\ 2t}{2\ \left[\pi - \varepsilon(t)\right]\ \cos 2\varepsilon(t) - \sin 2\varepsilon(t)}$$

$\to t_{crit} \in (0.839393459, 0.839393461),$

$\to s_{crit} \in (2.665799044, 2.665799069),$

$\to E_{crit} \in (6.401903165, 6.401903294).$

Table 1:

iteration N	$Z_{crit}^{(lower)}$	$Z_{crit}^{(upper)}$
0	4.299	4.663
2	4.4614	4.4857
4	4.47431	4.47601
6	4.475239	4.475357
8	4.47530381	4.4753119
10	4.475308262	4.475308823
12	4.475308560	4.475308614

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XI.

SIX channels

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$L = 3$

21 free parameters

Hermitian R and a weak symmetry,

skipped

$L = 1$ or $L = 5$:

$$\mathbf{r}_{(6,1)}^{(permuted)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$\mathbf{A}$ = asymmetric:

$$\mathbf{A}^{(permuted)}_{(interaction)} = \left(\begin{array}{cc|cc|cc} Z & Y & G & B & F & B \\ X & Z & C & F & C & G \\ \hline F & B & Z & Y & G & B \\ C & G & X & Z & C & F \\ \hline G & B & F & B & Z & Y \\ C & F & C & G & X & Z \end{array}\right).$$

eigenvalues = roots of quadratic equations

two = non-degenerate

two = doubly degenerate

$L = 2$ or $L = 4$:

$$\mathbf{r}_{(6,2)}^{(permuted)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$\mathbf{A}$ = symmetric:

$$\mathbf{A}^{(permuted)}_{(interaction)} = \left(\begin{array}{ccc|ccc} Z & X & X & C & D & G \\ X & Z & X & G & C & D \\ X & X & Z & D & G & C \\ \hline C & G & D & A & B & B \\ D & C & G & B & A & B \\ G & D & C & B & B & A \end{array}\right).$$

eigenvalues = roots of quadratic equations

two = non-degenerate

two = doubly degenerate

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XII.

$2M-1$ channels with $M=4$ etc

$M = 4$: four free parameters at all L:

$$\mathbf{A}_{(interaction)} = \begin{pmatrix} Z & X & Y & D & D & Y & X \\ X & Z & X & Y & D & D & Y \\ Y & X & Z & X & Y & D & D \\ D & Y & X & Z & X & Y & D \\ D & D & Y & X & Z & X & Y \\ Y & D & D & Y & X & Z & X \\ X & Y & D & D & Y & X & Z \end{pmatrix}.$$

Cardano formulae.

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XIII.

$2M$ channels with $M = 4$ etc

37 free parameters for $(2M, L) = (8, 4)$

(29 in pairs),

16 free parameters for $(2M, L) = (8, 2)$

(all in quadruplets),

8 free parameters for $(2M, L) = (8, 1)$ etc,

(all in octuuplets).

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XIV. The summary of the talk:

(1) Our particular recipe $\mathcal{P} \to \mathcal{R}$ allowing **finite rotations** proved feasible.

(2) Models may be useful as carrying **new nontrivial symmetries**.

(3) Coupled-channel Hamiltonians shown **equally** appealing within PTSQM as they were in Hermitian models.

(4) One feels encouraged to search for some **further** extensions of "quantum practice" in **quasi-Hermitian** directions.